2 (a) Starting from the equation of motion in a frame rotating uniformly with angular velocity $\mathbf{\Omega}$,

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p - \nabla\psi - 2\mathbf{\Omega}\times\mathbf{u} - \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}),$$

where $\mathrm{d}/\mathrm{d}t$ denotes the material derivative in the rotating frame, derive the equation for hydrostatic equilibrium of the fluid in the form

$$\frac{1}{\rho}\nabla p = -\nabla\Phi,$$

where the total effective potential Φ should be defined and its form carefully derived. Explain why, in the equilibrium, pressure p is constant, and also density ρ is constant, at any surface where Φ is constant.

(b) Explain why Φ is constant over the surface of a rotating fluid body. Show further that for a slow rotation, the relative difference $\Delta R/R$ between the equatorial radius $R + \Delta R$ and the polar radius R can be estimated as

$$\frac{\Delta R}{R} = \frac{1}{2}\frac{\Omega^2 R^3}{GM}.$$

Evaluate the magnitude of this relative difference for Jupiter, which has rotation period 10 hours, radius 0.1 solar radii, and mass 10^{-3} solar masses [you are referred to the solar data given on page 1 of this paper].

Explain, qualitatively, how the last result will be modified if we take into account a distortion of the spherically-symmetric gravitational field induced by the rotational distortion of the mass distribution inside the planet.

(c) A binary star system consists of two stars with masses M_1 and M_2 separated by a distance a, orbiting each other in circular orbits. Show that the period $2\pi/\Omega$ of the system is such that

$$\Omega^2 = \frac{G(M_1 + M_2)}{a^3}.$$

Assuming that the gravitational potential of each star can be approximated by that of a point mass at the centre of the star, and taking Cartesian coordinates such that the x-axis runs through the centres of the two stars and the orbits are in the $z = 0$ plane, derive the Roche form of the potential Φ. Sketch contours of constant Φ in the $z = 0$ plane, and explain the significance of the Roche lobes.

[Turn page]

3 A gaseous configuration moves under its internal pressure and self-gravity. Suppose that $p = p_0 + p'$, $\rho = \rho_0 + \rho'$ and $\psi = \psi_0 + \psi'$, where p', ρ' and ψ' are the perturbations to pressure, density and gravitational potential respectively, and p_0, ρ_0 and ψ_0 are equilibrium quantities. The perturbations and velocity $\mathbf{u}$ are small, so that quadratic and higher order expressions involving them may be neglected. By linearizing the basic fluid equations show that

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p' - \rho' \nabla \psi_0 - \rho_0 \nabla \psi',$$

$$\frac{\partial \rho'}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{u}),$$

$$\nabla^2 \psi' = 4\pi G \rho',$$

and, in the adiabatic approximation,

$$\frac{\partial p'}{\partial t} + \mathbf{u} \cdot \nabla p_0 = \Gamma_1 \frac{p_0}{\rho_0} \left(\frac{\partial \rho'}{\partial t} + \mathbf{u} \cdot \nabla \rho_0 \right).$$

Now prove that the linear equations of adiabatic radial stellar oscillations can be written as

$$\frac{dU}{dr} = \left(\frac{g_0}{c^2} - \frac{2}{r} \right) U - \frac{1}{\rho_0 c^2} p_1,$$

$$\frac{dp_1}{dr} = \left(\omega^2 - N^2 + 4\pi G \rho_0 \right) \rho_0 U - \frac{g_0}{c^2} p_1,$$

where $U(r)$ and $p_1(r)$ describe radial displacements $\delta\mathbf{r}$ and Eulerian pressure perturbations p' through

$$\delta \mathbf{r} = \hat{\mathbf{r}} U(r) \exp(i\omega t) \quad \text{and} \quad p' = p_1(r) \exp(i\omega t).$$

You should assume perturbations of the form $\rho' = \rho_1(r)\exp(i\omega t)$ and $\psi' = \psi_1(r)\exp(i\omega t)$. ω is the angular frequency, $\mathbf{u} = i\omega\delta\mathbf{r}$, $\hat{\mathbf{r}}$ is unit vector along r, c is adiabatic sound speed, g_0 is equilibrium gravitational acceleration, and N is Brunt-Vâisâlâ frequency. You may use the relations

$$c^2 = \Gamma_1 \frac{p_0}{\rho_0}, \quad N^2 = -g_0 \left(\frac{d \ln \rho_0}{dr} - \frac{1}{\Gamma_1} \frac{d \ln p_0}{dr} \right).$$

[**Hint:** By comparing the continuity and the Poisson equations, show that $d\psi_1/dr = -4\pi G \rho_0 U$].

Give brief mathematical and physical explanations why these equations are not adequate for describing the excitation and damping of stellar oscillations.

[Turn page]

4 A fluid has motion and variations only in one spatial direction x. By appropriately combining the momentum and continuity equations in their standard form (with no external forces), and the adiabatic energy equation which you may assume in the form

$$\rho\frac{DU}{Dt} = \frac{p}{\rho}\frac{D\rho}{Dt} ,$$

U being the internal energy per unit mass, derive the momentum and energy equations in conservative form:

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) = 0 ,$$

$$\frac{\partial}{\partial t}\left(\rho U + \frac{1}{2}\rho u^2\right) + \frac{\partial}{\partial x}\left(\rho u\left(U + \frac{p}{\rho} + \frac{1}{2}u^2\right)\right) = 0 .$$

Hence deduce the jump conditions for a steady shock:

$$\begin{aligned} \rho_1 u_1 &= \rho_2 u_2 \\ \rho_1 u_1^2 + p_1 &= \rho_2 u_2^2 + p_2 \\ U_1 + \frac{p_1}{\rho_1} + \frac{1}{2}u_1^2 &= U_2 + \frac{p_2}{\rho_2} + \frac{1}{2}u_2^2 \end{aligned}$$

where subscripts 1 and 2 denote conditions just upstream and just downstream of the shock.

For an ideal gas, $U = (\Gamma - 1)^{-1}p/\rho$, where Γ is the adiabatic exponent. Show that for a strong shock, for which the upstream Mach number $M_1 \gg 1$, the jump conditions imply

$$\begin{aligned} \frac{\rho_2}{\rho_1} &= \frac{\Gamma+1}{\Gamma-1} = \frac{u_1}{u_2} \\ \frac{p_2}{p_1} &= \frac{2\Gamma M_1^2}{\Gamma+1} . \end{aligned}$$

Consider the following highly simplified model based on the above. The Sun loses mass at a rate $\dot{M}$ per unit time in the form of the solar wind. At the orbit of the earth the wind is supersonic and the measured velocity is u_E. At a greater distance r_s from the Sun, the wind encounters a strong stationary shock. At an even greater distance, r_h, the wind encounters the heliopause, the boundary between the solar wind and the interstellar medium (which are assumed not to interpenetrate). Assume that the speed in the wind is constant in the supersonic regime, and $\Gamma = 5/3$. Deduce that the shock is located at

$$r_s = \left(\frac{\dot{M}}{\pi u_E \rho_2}\right)^{1/2} .$$

Between the shock and the heliopause, ρ and $\rho u^2 + p$ are essentially constant, and the speed u declines rapidly with distance from the Sun, so that $\rho u^2 + p = p_x$, where p_x is the pressure in the interstellar medium. Show that

$$r_s = \left(\frac{\dot{M}u_E}{4\pi p_x}\right)^{1/2} .$$

Assuming that $\dot{M}$ is constant over the Sun's main-sequence lifetime, show that the location of the heliopause varies slowly with time as

$$r_h \propto t^{1/3} .$$

Taking $\dot{M} = 10^{-13}M_\odot$ per year, $u_E = 4 \times 10^5$ m s^{-1}, the age of the Sun to be 5×10^9 years, and p_x to be 10^{-14} N m^{-2} (appropriate to a temperature of 10^4K and a density of 10^5 atoms m^{-3}), estimate the present positions of the shock and the heliopause.

[Turn page]

5 Write briefly on **two** of the following topics:

(a) differential rotation and meridional circulation in stars;

(b) solar seismology;

(c) formation of supersonic flows in compressible fluids;

(d) nonlinear acoustic waves and shocks.

[End of paper]
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