Queen Mary
UNIVERSITY OF LONDON

M.Sc. Astrophysics

ASTM112 Astrophysical Fluid Dynamics

Duration: 3 hours
17 May 2006 18:15

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best **3** *questions answered will be counted.*

Calculators ARE permitted in this examination.

Notation

The following notation is used throughout unless otherwise stated. The pressure, density, gravitational potential and adiabatic exponents are denoted by p, ρ, ψ, Γ_1 and Γ_3 respectively. The equilibrium values of these quantities are sometimes distinguished using a zero subscript. The position vector is denoted by $\mathbf{r}$ or $\mathbf{x}$, the time by t, the velocity by $\mathbf{u}$, the surface radius of a spherical configuration by R, and the gravitational constant by G. Vectors are denoted by boldface type.

Astronomical and Physical Data

Mass of the Sun	$M_\odot$	$2.0 \times 10^{30}\,\mathrm{kg}$
Surface radius of the Sun	$R_\odot$	$7.0 \times 10^{8}\,\mathrm{m}$
Luminosity of the Sun	$L_\odot$	$3.8 \times 10^{26}\,\mathrm{J\,s^{-1}}$
Gravitational constant	G	$6.67 \times 10^{-11}\,\mathrm{kg^{-1}m^3s^{-2}}$
Speed of light in a vacuum	c	$3.0 \times 10^{8}\,\mathrm{m\,s^{-1}}$

Standard Formulae

Candidates may assume the following set of basic equations and formulae:

In spherical polar coordinates (r, θ, ϕ)

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)$$

and

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\,.$$

For $\mathbf{u} = (u_r, u_\theta, u_\phi)$,

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$$

[Turn page]

and

$$\nabla\times\mathbf{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ u_r & ru_\theta & r\sin\theta u_\phi \end{vmatrix} .$$

The spherical harmonic $Y_l^m(\theta,\phi) = P_l^{|m|}(\cos\theta)\exp(im\phi)$, where $P_l^{|m|}$ denotes the associated Legendre function, satisfies the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_l^m}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_l^m}{\partial\phi^2} + l(l+1)Y_l^m = 0 ,$$

where l is a non-negative integer and m is an integer such that $|m| \leq l$. Further

$$\nabla^2(Y_l^m r^l) = 0 \qquad \nabla^2(Y_l^m r^{-l-1}) = 0 .$$

In cylindrical polar coordinates (r,ϕ,z), with $\mathbf{u} = (u_r, u_\phi, u_z)$,

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z} , \quad \nabla\times\mathbf{u} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ u_r & ru_\phi & u_z \end{vmatrix} .$$

The material derivative is given by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}.\nabla .$$

The equation of motion for an inviscid fluid may be assumed in the form

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p - \rho\nabla\psi .$$

The continuity equation may be assumed in the form

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 .$$

The energy equation may be assumed in the form

$$\frac{Dp}{Dt} - \frac{\Gamma_1 p}{\rho}\frac{D\rho}{Dt} = \rho(\Gamma_3 - 1)\left(\epsilon - \frac{1}{\rho}\nabla\cdot\mathbf{F}\right) ,$$

where ϵ is the heat generated per unit mass, and $\mathbf{F}$ is the heat flux. For adiabatic motion, the right-hand side of this equation is zero.

The gravitational potential satisfies Poisson's equation, $\nabla^2\psi = 4\pi G\rho$, which may be assumed to have the solution

$$\psi(\mathbf{r},t) = -\int_V \frac{G\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V' ,$$

where the integration is taken over the fluid volume V, and $\mathrm{d}V'$ denotes the volume element $\mathrm{d}^3\mathbf{r}'$.

[Turn page]

1 A compressible fluid is stratified in uniform gravitational field, with gravitational acceleration g_0 directed downwards. Equilibrium pressure $p_0(z)$ and density $\rho_0(z)$ depend on vertical coordinate z only; the axis z points upwards.

(a) Show that in hydrostatic equilibrium,

$$\frac{dp_0(z)}{dz} = -\rho_0(z) g_0.$$

Explain the physical nature of this hydrostatic relation.

(b) Consider a small vertical displacement of a small fluid element from its equilibrium position at $z = 0$. Assume that the fluid element is always in pressure equilibrium with surrounding fluid, and the surrounding fluid remains undistorted.

Show that in the adiabatic approximation (i.e. discarding any heat exchange between the fluid element and the surrounding fluid), the small variations of pressure δp and density $\delta\rho$ in the fluid element satisfy

$$\delta p = \Gamma_1 \frac{p_0(0)}{\rho_0(0)} \delta\rho,$$

where Γ_1 is the adiabatic exponent. Discuss briefly, in which circumstances the adiabatic approximation is applicable for describing astrophysical fluids.

(c) Show that after such a displacement by an amount δz in vertical direction, the density in the fluid element will differ from the density of the fluid which is around it by an amount

$$\rho' = \frac{\rho_0(0)}{g_0} N^2(0)\, \delta z,$$

where

$$N^2 = -g_0 \left(\frac{d \ln \rho_0}{dz} - \frac{1}{\Gamma_1} \frac{d \ln p_0}{dz} \right).$$

Explain, qualitatively, how the motion of the fluid element will develop after this initial displacement, when (i) $N^2 > 0$ and (ii) $N^2 < 0$.

(d) Show that linearized equation of motion of this particular fluid element can be written as

$$\rho_0(0) \frac{d^2 \delta z}{dt^2} = -\rho' g_0 = -\rho_0(0) N^2(0)\, \delta z.$$

Show further that the solution to this equation is $\delta z(t) = A \exp(i\omega t)$, with $\omega^2 = N^2$. When $N^2 > 0$, what is the restoring force of the resulted oscillatory solution? When $N^2 < 0$, you have a solution which grows exponentially with time. Discuss briefly the physical nature of this solution. Why our analysis is not applicable for predicting correctly the development of this solution at large t?

[Turn page]