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M.Sc. Examinations 2005

ASTM109/MAS415 Stellar Structure and Evolution

27th May 2005 18.15 - 21.15; Duration 3 hrs 0 min

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators may NOT be used in this examination.

In all questions: M is the mass, M_r the mass interior to radius r, R is the radius, L the luminosity and T_{eff} the effective temperature of a star. P, ρ and T denote the pressure, density and temperature respectively. κ is the opacity per unit mass, ϵ the rate of energy production per unit mass and μ denotes the mean molecular weight. c_p, c_v are the specific heats at constant pressure and volume, $\gamma = c_p/c_v$ and $\mathcal{R}$ is the gas constant where $\mathcal{R} = \mu(c_p - c_v)$.

$L = 4\pi R^2 F_{Rad}$ and F_{Rad} is given by

$$F_{Rad} = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{dT}{dr}.$$

c, G, a, σ are respectively the velocity of light, the constant of gravity, the Stefan-Boltzmann radiation constant and Stefan's constant. X, Y, Z are the mass fractions respectively of hydrogen, helium and the heavier elements.

The central density ρ_c, central temperature T_c and central pressure P_c of a polytrope of index n are

$$\rho_c = a_n\frac{M}{R^3}, \qquad T_c = b_n\frac{\mu GM}{\mathcal{R}R}, \qquad P_c = c_n\frac{GM^2}{R^4}.$$

SECTION A. You should attempt all questions. Marks awarded are shown next to the question.

Question A1. 12 marks. A spherical gas cloud of mass M and initial radius r_o is pressure-free. It collapses freely under its own gravity. Show that the free-fall time τ is given by

$$\tau^2 = \frac{\pi^2 {r_o}^3}{8GM}.$$

You may assume that

$$\int_0^{r_o} \frac{dr}{\sqrt{r_o/r - 1}} = \pi r_o/2.$$

A proto-star of mass $M_\odot/10$ starts a free-fall collapse phase at an effective temperature equal to the current solar value and a luminosity of $100L_\odot$. Calculate its FREE FALL time in terms of that of the Sun.

Question A2. 6 marks. Show that for a fully ionized gas consisting of atomic hydrogen, helium and ^{12}C only, the mean molecular weight μ is given by

$$\mu = \frac{12}{(7 + 17X + 2Y)}.$$

In an evolved star, $Y = 0.25$ while Z has increased to 0.25, calculate the mean molecular weight for this star.

Question A3. 12 marks. By considering the forces acting on a volume element, show that for a spherically symmetric star to be in hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}.$$

In a particular hypothetical star, $\rho = \rho_c(1 - r^2/R^2)$. Find the value of ρ_c in terms of M and R.
Given that P is zero when $r = R$, show that

$$P_c = \frac{15GM^2}{16\pi R^4}.$$

Question A4. 6 marks. The Lane-Emden equation in the usual notation is

$$\frac{1}{\xi^2}\frac{d}{d\xi}[\xi^2\frac{d\theta}{d\xi}] = -\theta^n.$$

Find the solution to the Lane-Emden equation when $n = 0$.

Find the value of ξ at the surface of a star that behaves like a polytrope with index 0.

Question A5. 12 marks. A group of homogeneous stars, composed of an ideal gas, all have the same chemical composition. All the energy is carried by radiation and the main opacity is electron scattering so that κ is a constant. Energy generation is by the CN cycle, with $\epsilon = \epsilon_0 \rho T^{17}$.

Show that

$$R \sim M^{0.8}$$

$$L \sim M^3.$$

Determine the slope in the H-R diagram of the line that these stars will lie on.

Question A6. 12 marks. Near the centre of an evolved star, there is an isothermal helium core of mass M_s and radius r_s. The gravitational energy of the core is given by

$$V = -\frac{3GM_s^2}{5r_s}.$$

Show that there is a maximum value for the pressure at the surface of the core given when

$$r_s = \frac{4\mu_s GM_s}{15\mathcal{R}T}.$$

Please turn over for Section B questions

SECTION B Each question carrries 40 marks. You may attempt all questions but only the best ONE will be counted.

Question B1. A group of homogeneous stars, composed of an ideal gas, all have the same chemical composition and polytropic index. The opacity is given by $\kappa = \kappa_0 \rho T^{-3.5}$. Assuming that all the energy is transferred by radiation, show that

$$L \propto M^{5.5} R^{-0.5}.$$

Given that U is the internal energy of a star and V the gravitational energy, obtain the virial theorem

$$3(\gamma - 1)U + V = 0.$$

The gravitational potential energy of a star of polytropic index n is given by

$$\frac{-3GM^2}{(5-n)R}.$$

In a particular star, there are no nuclear energy sources, the polytropic index is 3 and $\gamma = 5/3$. Show that a star like the above will move along a line in the HR diagram with a slope of 0.8. Show that t, the time taken by such a star to evolve from a large radius to some smaller radius R_0, is given by

$$t \propto M^{-3.5} R_0^{-0.5}.$$

A group of stars of the same type, but different masses were all formed at the same time. Find the slope of the line in the HR diagram that these stars will lie on at a given time.

In fact, prior to the radiative stage, such stars are convective so that the polytropic index is 3/2. They also evolve with a constant T_{eff}. Obtain the time taken by such a star of mass M to contract from a large radius to a radius R_1.

Question B2. Derive the Schwarzschild condition for the onset of convection in an ideal gas, namely

$$\frac{TdP}{PdT} \leq \frac{\gamma}{\gamma - 1},$$

where γ is the ratio of specific heats, T the temperature and P the pressure.

Define the optical depth, τ, in a stellar atmosphere.

In the atmosphere of a star at optical depth τ,

$$T^4 = (3/4)T_{eff}^4(\tau + 2/3).$$

Show that

$$\frac{1}{T}\frac{dT}{d\tau} = \frac{1}{4(\tau + 2/3)}.$$

The opacity is given by

$$\kappa = \kappa_1 P T^4.$$

Assuming that the atmospere is an ideal gas and is in hydrostatic equilibrium and that the mass and thickness of the atmosphere are both negligible compared to the mass and radius of the star, find an expression for P in terms of τ within the atmosphere given that $P = 0$ at $\tau = 0$.

Given that γ has a value of 5/3, calculate the value of τ at the point where convection sets in.

Question B2 continues on the next page

You may assume that the mass of the convective zone is also small, so that M_r is the same as M. Also, in the convection zone it may be assumed that $P = KT^{5/2}$.

Show that at a depth h inside the zone, the temperature is approximately given by

$$T - T_s = \frac{2GM\mu h}{5\mathcal{R}R^2},$$

where T_s is the temperature at the top of the convective zone.

Question B3. State the Exclusion Principle.

In a degenerate gas all states are filled up to a threshold momentum p_0 and none above. Show that the electron number density is

$$N_e = \frac{8\pi}{3h^3}{p_0}^3,$$

where h is Planck's constant.

Show that when the electrons are moving with speeds small compared to the speed of light,

$$P = \frac{8\pi}{15h^3 m_e}{p_0}^5,$$

where m_e is the electron mass.

Show that for a white dwarf, assumed to be made of non-relativistic degenerate material, the mass M and radius R satisfy the relationship

$$MR^3 = A,$$

where A is a constant.

Find the corresponding expression for P when the electrons are assumed to be moving at speeds close to that of light (i.e relativistic). Show that in this case there is only one possible mass for the star.

A star has a non-relativistic degenerate helium core of mass M_c and radius R_c surrounded by a hydrogen burning shell. The mass radius relationship for this core may be assumed to be like that of a white dwarf. The energy generated by fusion of a mass m of hydrogen to helium is approximately $0.02mc^2$, where c is the speed of light, so that for a star with luminosity L, additional degenerate mass is deposited uniformly and gently on the core at a rate

$$\frac{dM_c}{dt} = \frac{50L}{c^2}.$$

Show that the rate of release of gravitational energy in the core is

$$\frac{100GM_cL}{c^2R_c}.$$

You may assume that the gravitational potential energy of the core is

$$\frac{-6G{M_c}^2}{7R_c}.$$

END OF QUESTION PAPER