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M.Sc. Examinations 2006

ASTM109 Stellar Structure and Evolution

23rd May 2006 10.00 - 13.00; Duration 3 hrs 0 min

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators may be used in this examination.

In all questions: M is the mass, M_r the mass interior to radius r, R is the radius, L the luminosity and T_{eff} the effective temperature of a star. P, ρ and T denote the pressure, density and temperature respectively. κ is the opacity per unit mass, ϵ the rate of energy production per unit mass and μ denotes the mean molecular weight. c_p, c_v are the specific heats at constant pressure and volume, $\gamma = c_p/c_v$ and $\mathcal{R}$ is the gas constant where $\mathcal{R} = \mu(c_p - c_v)$.

$L = 4\pi R^2 F_{Rad}$ and F_{Rad} is given by

$$F_{Rad} = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{dT}{dr}.$$

c, G, a, σ are respectively the velocity of light, the constant of gravity, the Stefan-Bolzmann radiation constant and Stefan's constant. X, Y, Z are the mass fractions respectively of hydrogen, helium and the heavier elements.

The central density ρ_c, central temperature T_c and central pressure P_c of a polytrope of index n are

$$\rho_c = a_n\frac{M}{R^3}, \qquad T_c = b_n\frac{\mu GM}{\mathcal{R}R}, \qquad P_c = c_n\frac{GM^2}{R^4}$$

SECTION A. You should attempt all questions. Marks awarded are shown next to the question.

Question A1. 6 marks Show that for a fully ionized gas consisting of atomic hydrogen, helium and ^{14}N nitrogen only, the mean molecular weight μ is given by

$$\mu = \frac{28}{(16 + 40X + 5Y)}.$$

In a particular star, $X = 0.7$ and $Z = 0.02$, calculate μ.

Question A2. 10 marks Derive the Schwarzschild condition for the onset of convection

$$\frac{PdT}{TdP} \geq \frac{\gamma - 1}{\gamma},$$

where γ is the ratio of specific heats.

Question A3. 12 marks A star is in hydrostatic equilibrium so that $\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}$. The gravitational energy is given by

$$V = -\int_0^R \frac{GM_r dM_r}{r}.$$

Show that if the pressure at the surface is P_s,

$$V = 4\pi R^3 P_s - 3\int_0^R 4\pi r^2 P dr.$$

Show also that the potential energy of a star where $\rho = CT^n$ and $P_s = 0$ is given by

$$\frac{-3GM^2}{(5-n)R}.$$

Question A4. 12 marks A group of homogeneous stars, composed of an ideal gas, all have the same chemical composition. The opacity is given by $\kappa = \kappa_0 \rho T^{-4}$, and energy generation is by the CN cycle, with $\epsilon = \epsilon_0 \rho T^{16}$.

Show that

$$R^3 \sim M^2$$

$$L^3 \sim M^{16}.$$

Sketch the line in the H-R diagram that these stars will lie on, indicating its slope.

In terms of mass and temperature, what types of stars are these likely to be?

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Question A5. 12 marks State the Exclusion Principle.

In a degenerate gas all states are filled up to a threshold momentum p_0 and none above. Show that the electron number density is

$$N_e = \frac{8\pi}{3h^3}{p_0}^3,$$

where h is Planck's constant.

Show that when the electrons are moving with speeds small compared to the speed of light,

$$P = \frac{8\pi}{15h^3 m_e}{p_0}^5,$$

where m_e is the electron mass.

Show that for a white dwarf, assumed to be made of non-relativistic degenerate material, the mass M and radius R satisfy the relationship

$$MR^3 = A,$$

where A is a constant.

Question A6 8 marks Near the centre of an evolved star, there is an isothermal helium core of mass M_s and radius r_s. Show that there is a maximum value for the pressure, P_s at the surface of the core given when

$$r_s = \frac{2\mu_s G M_s}{3\mathcal{R}T}.$$

You may assume that

$$V = 4\pi {r_s}^3 P_s - 3\int_0^{r_s} 4\pi r^2 P dr,$$

and that

$$V = \frac{-3GM^2}{(2)r_s}.$$

Obtain an expression for this maximum pressure.

Please turn over for Section B questions

SECTION B Each question carrries 40 marks. Marks for each part are shown thus [x marks]. You may attempt all questions but only the best ONE will be counted.

Question B1 Determine the force per unit mass due to gravity of a sperical shell of radius r, thickness dr and density ρ at a point a distance x from the centre of the shell when

(a) $x > r$

(b) $x < r$.

Hence, or otherwise, show that the force per unit mass due to gravity at a point a distance r from the centre of a spherically symmetric star is given by $\frac{-GM_r}{r^2}$, where M_r is the mass contained inside the sphere of radius r.

[10 marks].

By considering the forces acting on a volume element, show that for a spherically symmetric star to be in hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}.$$

[5 marks].

Material in the star satisfies the ideal gas equation $P = \frac{\mathcal{R}}{\mu}\rho T$ and behaves like a polytrope of index n so that $\rho = CT^n$.

Derive the Lane-Emden equation in the usual notation as

$$\frac{1}{\xi^2}\frac{d}{d\xi}[\xi^2\frac{d\theta}{d\xi}] = -\theta^n.$$

[10 marks]

An exact solution to the Lane-Emden equation is given by

$$\theta = 1 - \frac{\xi^2}{6}.$$

Deduce the value of n.

Deduce also the value of ξ at the surface of a star that behaves like such a polytrope.

Find in terms of the radius of the star, the distance from the centre at which the temperature is 75% of the central temperature and the temperature in terms of the central temperature at a distance from the centre of 80% of the radius.

[15 marks].

Question B2. A group of homogeneous stars, composed of an ideal gas, all have the same chemical composition and polytropic index. The opacity is given by $\kappa = \kappa_o\rho^{4/3}T^{-4.5}$. Assuming that all the energy is transfered by radiation, show that

$$L \propto M^{37/6}R^{-0.5}.$$

[5 marks].

Given that U is the internal energy of a star and V the gravitational energy, show that

$$3(\gamma - 1)U + V = 0.$$

You may assume that

$$V = -3\int_0^R 4\pi r^2 P dr.$$

[5 marks].

The gravitational potential energy of a star of polytropic index n is given by

$$\frac{-3GM^2}{(5-n)R}.$$

In the stars above, γ is 5/3, while the polytropic index is 3. Assuming that there are no nuclear energy sources, show that t_o, the time taken by such a star to evolve from a large radius to some smaller radius R_o, is given by

$$t_o = \frac{3GM^2}{2R_oL_o}.$$

Find the slope of the line in the H-R diagram along which stars, behaving like the above star, will evolve.

[15 marks].

The star was in fact convective with a polytropic index of 1.5. Find, in terms of t_o, the time taken by the star in adjusting internally from being convective to being radiative. You may assume that both the luminosity and effective temperature remained constant during this adjustment stage.

[10 marks].

Sketch the paths taken in the H-R diagram of a pre-main sequence star of about one solar mass

[5 marks].

Question B3 Define the optical depth, τ, in a stellar atmosphere.

Starting from the equation for radiative flux, F, given in the rubric, show that in the atmosphere of a star

$$\frac{ac}{3}T^4 = F(\tau + B),$$

where B is a constant of integration. At a large distance from the star all the radiation is moving outwards so that $F = \frac{ac}{2}T^4$. Show that

$$T^4 = (3/4)T_{eff}^4(\tau + 2/3),$$

where $\frac{ac}{4}T_{eff}^4 = F$.

[6 marks].

The opacity is given by

$$\kappa = \kappa_1 P^{0.5}T^8.$$

Assuming that the atmospere is in hydrostatic equilibrium and that the mass and thickness of the atmosphere is negligible compared to that of the star, find an expression for P in terms of τ within the atmosphere given that $P = 0$ at $\tau = 0$.

The Schwarzschild condition for the onset of convection is

$$\frac{TdP}{PdT} \leq \frac{\gamma}{\gamma - 1},$$

where γ is the ratio of specific heats, Find the value of τ when convection sets in assuming that γ has a value of 5/3.

[14 marks].

In the convection zone it may be assumed that $P = KT^{5/2}$ and that the mass in the convective zone is small, so that here M_r is the same as M.

Show that at a depth h inside the zone, the temperature is approximately given by

$$T = \frac{2GM\mu h}{5\mathcal{R}R^2}.$$

assuming that the surface temperature is much less than T.

[10 marks].

In the radiative central regions, the opacity is given by

$$\kappa = \kappa_0 \rho T^{-3.5}.$$

By considering the conditions at the boundary between the convective and radiative regions, show that T_b, the temperature at the bottom of the convective layer behaves like $L^{2/7}$.

[10 marks].

END OF QUESTION PAPER