

M. Sci. Examination by course unit 2009

MTH703U Advanced Cosmology

Duration: 3 hours

Date and time: 5th May 2009, 1430-1730

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best 4 questions answered will be counted.

Calculators ARE permitted in this examination. The unauthorized use of material stored in pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): J. E. Lidsey

You are reminded of the following information, which you may use without proof:

The following constants may be assumed:

$$\text{Speed of light, } c = 3.0 \times 10^8 \text{ m s}^{-1}$$

$$\text{Gravitational constant, } G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$$

$$\text{Boltzmann's constant, } k_B = 1.38 \times 10^{-23} \text{ JK}^{-1}$$

$$\text{Radiation constant, } \alpha = 7.565 \times 10^{-16} \text{ J m}^{-3} \text{ K}^{-4}$$

$$\text{Proton mass-energy, } m_p c^2 = 938.3 \text{ MeV}$$

$$\text{Neutron mass-energy, } m_n c^2 = 939.6 \text{ MeV}$$

$$\text{Mega Parsec, } 1 \text{ Mpc} = 3.09 \times 10^{22} \text{ m}$$

$$\text{Hubble time, } H_0^{-1} = 9.8 \times 10^9 h^{-1} \text{ yr} = 3.09 \times 10^{17} h^{-1} \text{ s}$$

$$\text{The Conversion Factor, } 1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$$

$$\text{The Conversion Factor, } 1 \text{ J} = 1 \text{ kg m}^2 \text{ s}^{-2}$$

The following formulae may be assumed:

Friedmann Equation

$$H^2 = \frac{8\pi G}{3}\rho + \frac{8\pi G}{3}\Lambda - \frac{kc^2}{a^2},$$

where $H = \dot{a}/a$ is the Hubble parameter, a is the scale factor of the universe, ρ is the mass density, Λ is the cosmological constant, k is a constant and overdots denote time derivatives.

Conservation Equation

$$\dot{\rho} + 3H\left(\rho + \frac{p}{c^2}\right) = 0,$$

where p represents the pressure of the matter in the universe.

Acceleration Equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right).$$

Question 1

- (a) [4 marks] Give the definition of the Ω -parameter in terms of the critical density of the universe. Show that in terms of Ω , the Friedmann equation can be expressed in the form

$$\Omega - 1 = \frac{kc^2}{\dot{a}^2}.$$

- (b) [4 marks] A university professor develops a theory where the universe is dominated by a new form of matter with an equation of state $p = -\frac{2}{3}\rho c^2$. Starting from the conservation equation, derive how the density of this matter varies with the scale factor of the universe.
- (c) [5 marks] Assuming that the universe is spatially flat and dominated by the new form of matter described in part (b), show that the scale factor grows with time as $a \propto t^2$.
- (d) [6 marks] Suppose that the universe expands as $a \propto t^2$ for all values of Ω . Sketch how Ω would vary with time for the three cases where initially $\Omega > 1$, $\Omega = 1$ and $\Omega < 1$, respectively. (Explain your reasoning).
- (e) [6 marks] What is the flatness problem of the big bang theory? Explain how this problem could be solved if the universe did indeed expand as $a \propto t^2$ for a finite time in its very distant past. By what factor must the volume of the universe increase during this era to solve the flatness problem?

Question 2

- (a) [5 marks] Explain why the inverse of the Hubble constant, H_0^{-1} , provides an upper limit to the age of the universe when $\Lambda = 0$.
- (b) [4 marks] For a spatially flat, pressureless universe with vanishing cosmological constant, write down how the scale factor varies with time. Hence, derive an expression for the age of such a universe in terms of the Hubble constant.
- (c) [6 marks] Verify that the solution to the Friedmann equation for a positively-curved, pressureless universe with vanishing cosmological constant can be expressed in the parametric form

$$a(\theta) = \frac{4\pi G\rho_0 a_0^3}{3}(1 - \cos \theta), \quad t(\theta) = \frac{4\pi G\rho_0 a_0^3}{3}(\theta - \sin \theta),$$

where ρ_0 and a_0 represent the present-day values of the matter density and scale factor, respectively. (You may assume $kc^2 = +1$.)

- (d) [5 marks] Given the solution in part (c), derive an expression relating the age of a positively-curved, pressureless universe to its size, a_{\max} , at the moment when such a universe starts to recollapse. What is the lifetime of such a universe when expressed in terms of a_{\max} ?
- (e) [5 marks] Explain, briefly, why the age of a negatively-curved universe is greater than that of a spatially flat universe with the same value of the Hubble constant.

Question 3

- (a) [4 marks] Explain what is meant by isotropy and homogeneity in the universe.
- (b) [5 marks] Summarize the key properties of the cosmic microwave background radiation.
- (c) [5 marks] Describe, briefly, the key physical processes that resulted in the formation of the cosmic microwave background at the epoch of decoupling.
- (d) [6 marks] The horizon distance at a time t (corresponding to the furthest distance a photon could have travelled since the big bang) is given by

$$d_H(t) = a(t) \int_0^t \frac{c}{a(t)} dt.$$

Explain what is meant by the ‘rescaled horizon distance’. For a pressureless universe with critical density and vanishing cosmological constant, verify that the rescaled horizon distance corresponding to the decoupling era, $\tilde{d}_H(t_{\text{dec}})$, satisfies the condition

$$\tilde{d}_H(t_{\text{dec}}) \approx 0.1ct_0.$$

- (e) [5 marks] Explain why the approximate equality in part (d) is a problem for the big bang theory.

Question 4

- (a) [3 marks] State how the density of a relativistic particle species varies with the scale factor and temperature of the universe, respectively.
- (b) [4 marks] What is the physical reason why the density of radiation decreases more rapidly than that of non-relativistic matter as the universe expands?
- (c) [4 marks] Given that the universe was dominated by relativistic matter when it was one second old and had a temperature $T \approx 10^{10}\text{K}$ at that time, show that at some earlier time, t , the temperature was given by

$$\frac{T}{10^{10}\text{K}} \approx \sqrt{\frac{1\text{sec}}{t}}.$$

- (d) [4 marks] Estimate the age and temperature of the universe when it was the size of a golf ball.
- (e) [4 marks] Estimate the age and temperature of the universe when the neutrons started to behave non-relativistically.
- (f) [6 marks] A cosmologist predicts that non-relativistic, stable particles are formed when the universe was 10^{-15} seconds old. Assuming that the density of these particles at the time of formation was 10^{-6} times that of the critical density, estimate the age of the universe when these particles come to dominate the density. (You may assume that all other particle species in the universe were behaving relativistically.)

Question 5

- (a) [3 marks] Explain what is meant by cosmological redshift and state how it is related to the scale factor of the universe.
- (b) [5 marks] The proper distance to a galaxy with redshift z is given by the integral

$$D_P = c \int_0^z \frac{dz}{H(z)},$$

where $H(z)$ denotes the Hubble parameter as a function of z . Consider a spatially flat, pressureless universe with vanishing cosmological constant. Use the Friedmann equation to determine how the Hubble parameter varies with redshift and hence show that the proper distance to a galaxy with a redshift $z = 3$ is $D_P = c/H_0$.

- (c) [6 marks] Give the definition of the deceleration parameter. Show that in a negatively curved universe with a general equation of state $p = (\gamma - 1)\rho c^2$, where $0 \leq \gamma \leq 2$ is a constant, the deceleration parameter satisfies

$$q < \frac{3\gamma - 2}{2}.$$

- (d) [6 marks] Summarize the main observational evidence that the universe today is dominated by a non-zero cosmological constant with a density about 70% the critical density.
- (e) [5 marks] Prove that if the present density of pressureless matter is less than the critical density, the universe will never recollapse if such a cosmological constant does indeed exist.

Question 6

- (a) [6 marks] Describe, briefly, the three primary sources of anisotropy in the temperature of the cosmic microwave background radiation.
- (b) [5 marks] What is the main evidence that there is baryonic dark matter in the universe?
- (c) [5 marks] Describe the most important steps in the formation of primordial helium.
- (d) [4 marks] List the following events in chronological order: the formation of large-scale structure in the universe, the decoupling era, the end of inflation, the primordial nucleosynthesis era, the epoch of matter-radiation equality.
- (e) [5 marks] Outline, quantitatively, how the density perturbations in the dark matter evolve during the radiation- and matter-dominated eras. What restrictions must be satisfied by the density perturbation if galaxies are to have formed by the present time?

End of Paper