Queen Mary
University of London

M.Sc. EXAMINATION

ASTM108 Cosmology

4 May 2005
Time 10.00-13.00

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

Calculators ARE permitted in this examination. The unauthorised use of material stored in a pre–programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

The following constants may be assumed:

Speed of light, $c = 3.0 \times 10^8\,\mathrm{m\,s^{-1}}$

Gravitational constant, $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$

Boltzmann's constant, $k_B = 1.38 \times 10^{-23}\,\mathrm{JK^{-1}}$

Radiation constant, $\alpha = 7.565 \times 10^{-16}\,\mathrm{J\,m^{-3}\,K^{-4}}$

Proton mass–energy, $m_p c^2 = 938.3\,\mathrm{MeV}$

Neutron mass–energy, $m_n c^2 = 939.6\,\mathrm{MeV}$

Hubble time, $H_0^{-1} = 9.8 \times 10^9 h^{-1}\,\mathrm{yr} = 3.09 \times 10^{17} h^{-1}\,\mathrm{s}$

The Conversion Factor, $1\,\mathrm{eV} = 1.602 \times 10^{-19}\,\mathrm{J}$

The following formulae may be assumed:

Friedmann Equation

$$H^2 = \frac{8\pi G}{3}\rho + \frac{8\pi G}{3}\Lambda - \frac{kc^2}{a^2}$$

where $H = \dot{a}/a$ is the Hubble parameter, a is the scale factor of the universe, ρ is the mass density, Λ is the cosmological constant, k is a constant and overdots denote time derivatives.

Conservation Equation

$$\dot{\rho} + 3H\left(\rho + \frac{p}{c^2}\right) = 0$$

where p represents the pressure of the matter in the universe.

1. (a) State the Cosmological Principle.

(b) Show, with the aid of a simple diagram, that the isotropy of the universe implies homogeneity.

(c) State Hubble's law and show, also with the aid of a simple diagram, how Hubble's law follows as a direct consequence of the Cosmological Principle.

(d) For a universe containing only pressureless matter and no cosmological constant, show that the Friedmann equation can be written as

$$H(z) = H_0(1+z)(1+\Omega_0 z)^{1/2}$$

where $\Omega \equiv \rho/\rho_c$, ρ_c is the critical density, the relation between redshift and scale factor is $1+z = a_0/a$ and a subscript '0' denotes present-day values.

(e) Using your answer to part (d), find an expression that determines the value of the Ω–parameter at a given redshift in terms of z and Ω_0. Hence, estimate the value of Ω at the epoch of decoupling in terms of the present–day value Ω_0.

2. (a) By differentiating the Friedmann equation, derive the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{8\pi G}{3}\Lambda$$

(b) What is the definition of the deceleration parameter? For a spatially flat universe containing pressureless matter and a cosmological constant, derive an expression for the deceleration parameter in terms of the density of the matter relative to the critical density.

(c) Discuss, briefly, the main observational evidence that the present–day value of the deceleration parameter is negative. What is the best-fit value of q_0 to the observations?

(d) Discuss, briefly, how a cosmological constant increases the age of the universe for a given value of the Hubble constant.

(e) If the universe contains a cosmological constant at the present–time and $kc^2 = -1$, show by solving the Friedmann equation that the future destiny of our universe is to expand exponentially with time.

3. (a) Given that the energy density of radiation at a temperature T is $\epsilon_{\rm rad} = \alpha T^4$, where α is the radiation constant, estimate the energy density of the cosmic microwave background at the present time. Hence estimate the number density of photons in the universe today.

 [This question continues overleaf ...]

(b) Use your answer to part (a) to estimate the energy density and mass density of baryons at the present time.

(c) How does your answer to part (b) compare to the critical density of the universe today?

(d) Describe, briefly, the main observational evidence that some of the dark matter in the universe today is non–baryonic.

(e) Describe, briefly, the implications of a high photon-to-baryon ratio on the formation of the cosmic microwave background at the epoch of decoupling.

4. (a) In a universe containing only pressureless matter (and no cosmological constant), a sphere of density ρ and radius R evolves such that

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\rho\delta$$

By considering a slightly perturbed sphere with the same mass, but with radius R' and density $\rho' = \rho(1+\delta)$, where $\delta \ll 1$ is the density perturbation, show that δ evolves as

$$\ddot{\delta} + 2\frac{\dot{R}}{R}\dot{\delta} - 4\pi G\rho\delta = 0$$

(b) Starting from the Friedmann equation, and ignoring the effects of spatial curvature, show that the density of the sphere evolves as $\rho = 1/(6\pi G t^2)$. [You may assume the sphere expands as $R \propto t^{2/3}$].

(c) Hence, show that the density perturbation grows in proportion to the radius of the sphere, i.e., $\delta \propto R$.

(d) State the criterion that δ must satisfy for large-scale structure to have formed in the universe by the present time.

(e) Describe, briefly, the physical process that prevents structure from forming in a universe containing only ordinary (baryonic) matter until the epoch of decoupling. How does the presence of non–baryonic dark matter resolve this problem?

5. (a) Write down the equation of state for relativistic and non–relativistic particles, respectively.

(b) By integrating the conservation equation for each type of particle species, show that $\rho_{\rm rel}/\rho_{\rm nonrel} \propto 1/a$, where $\rho_{\rm rel}$ and $\rho_{\rm nonrel}$ are the densities of relativistic and non–relativistic particles, respectively, and a is the scale factor.

(c) Explain why the effects of spatial curvature in the universe may be neglected prior to the epoch of matter-radiation equality.

[This question continues overleaf ...]

(d) By solving the Friedmann equation, show that the age, t, of a universe dominated by relativistic particles is related to its temperature, T, by the relation

$$T\sqrt{t} = \text{constant}$$

Why is such a relation important for studying the evolution of the very early universe?

(e) Some theories of the early universe predict that non–relativistic particles are formed when the temperature is $T \approx 10^{28}$ K. Assuming the universe to be dominated by radiation at this time, calculate what the initial density of these particles would need to be, relative to the total density of the universe, if they are *not* to dominate the universe before the primordial nucleosynthesis era. [You may assume that the temperature of the universe is $T \approx 10^{10}$ K when it is one second old.]

6. (a) The Friedmann equation written in terms of the Ω–parameter has the form:

$$\Omega - 1 = \frac{kc^2}{a^2H^2}$$

Can the value of Ω ever become infinite? Explain your answer.

(b) Describe the flatness problem of the big bang model.

(c) Explain how a period of inflation is able to solve the flatness problem.

(d) Assuming that inflation occurred for a very brief interval and ended when the universe was about 10^{-34} seconds old, derive an expression relating the present-day value of Ω_0 to the value of Ω at the start of inflation and the ratio a_f/a_b, where a_b and a_f denote the scale factor at the start and end of inflation, respectively.

(e) Assuming typical values for the amount of inflation required to solve the flatness problems, use your answer to part (d) to estimate the range of values of Ω_0 predicted by inflation.

(f) Write a short note discussing the main observational evidence from the cosmic microwave background radiation that the value of Ω_0 today is close to unity.

[End of examination paper.]