Queen Mary
University of London

M.Sc. EXAMINATION

ASTM041 Relativistic Astrophysics

Monday, 9 May 2005
10:00-11:30

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in a pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

You are reminded of the following.

Physical Constants

Gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
Speed of light	c	3×10^{8} m s^{-1}
Solar mass	$M_\odot$	2×10^{30} kg
Solar radius	$R_\odot$	7×10^{5} km
1 kpc		3.09×10^{19} m

Notation

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.

Four-dimensional tensor indices are denoted by Latin letters $i, j, k, l, ...$ and take on the values 0, 1, 2, 3.

The metric signature (+ - - -) is used.

Useful formulae

The following results may be quoted without proof

Hamilton-Jacobi equation:

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} - m^2c^2 = 0,$$

where four-momentum $p_i = -\frac{\partial S}{\partial x^i}$ and $p_0 = E$ (energy), $p_3 = L$ (angular momentum).

Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2\left(\sin^2\theta d\phi^2 + d\theta^2\right).$$

Gravitational radius of a body of the mass M: $r_g = 2GM/c^2 = 3(M/M_\odot)$ km.

Kerr metric:

$$ds^2 = (1 - \frac{r_g r}{\rho^2})c^2dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2 +$$

$$\frac{2r_g rac}{\rho^2}\sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, $a = \frac{J}{Mc}$ and J is angular momentum.

For the Schwarzschild and Kerr metric: $x^0 = ct$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$.

SECTION A

Each question carries 20 marks. You should attempt ALL questions.

1. (a) A spacecraft exploring a planet of mass m and radius r moves around the planet along a circular orbit of radius $R = 6r$. Ignoring the transverse Doppler effect, evaluate the redshift z of the radio signal emitted by a probe left on the surface of the planet and received by the spacecraft.

(b) Another spacecraft moves very far from any gravitating bodies with acceleration a. The redshift of a photon emitted at the bottom of the rocket and detected at its top is $z' \approx 10^{-14}$. Evaluate the acceleration a if the height of the rocket is 100 m. [Hint: First calculate the gravitational redshift when $R - r \ll r$ and then apply the equivalence principle.]

2. (a) A star forms a black hole of mass M. Show that at the moment when the radius of the star is equal to $10^3 r_g$ its average density is

$$\rho \approx 2 \times 10^{10}\ \mathrm{kg\ m^{-3}} \left(\frac{M}{M_\odot}\right)^{-2}.$$

(b) Using simple Newtonian estimates, show that to an order of magnitude the radius of tidal disruption for a star of mass m and radius r in the gravitational field of a black hole of mass M is

$$R_{TD} \approx r\left(\frac{M}{m}\right)^{1/3}.$$

Compare this radius with the gravitational radius and find the black hole mass for which $R_{TD} = 10^3 r_g$. Give the answer in solar masses.

3. (a) An observer moves along a circular orbit of radius r in the equatorial plane ($\theta = \pi/2$) of a rotating black hole. If the gravitational field is described by the Kerr metric, show that this metric can be written in the form

$$ds^2 = \left(g_{00} - \frac{g_{03}^2}{g_{33}}\right)c^2dt^2 + g_{33}\left(d\phi - \Omega dt\right)^2,$$

where

$$\Omega = -\frac{g_{03}}{g_{33}} = \frac{r_g a}{(r^2 + a^2)r + r_g a^2}.$$

Use the Equivalence Principle to show that the corresponding non-inertial reference frame rotates with angular velocity Ω.

(b) Find the values of r corresponding to the limit of stationarity ($g_{00} = 0$) and the event horizon ($g_{11} = \infty$).

[Next section overleaf.]

SECTION B

Each question carries 40 marks. Only marks for the best ONE question will be counted.

1. (a) Consider the motion of a particle in the gravitational field of a Schwarzschild black hole. Using the Hamilton-Jacobi equation, show that

$$E\left(1-\frac{r_g}{r}\right)^{-1}\frac{dr}{dt}=c\sqrt{E^2-U_{eff}^2},$$

where U_{eff} is the "effective potential energy":

$$U_{eff}(r)=mc^2\sqrt{\left(1-\frac{r_g}{r}\right)\left(1+\frac{L^2}{m^2c^2r^2}\right)}.$$

Here L is the angular momentum and m is the mass of a particle.

(b) Explain how U_{eff} can be used to find stable and unstable circular orbits.

(c) Show that the radius of the stable circular orbit with angular momentum L is

$$r=\frac{L^2}{m^2c^2r_g}\left[1+\sqrt{1-\frac{3m^2c^2r_g^2}{L^2}}\right].$$

Evaluate the radius of the innermost stable circular orbit.

2. (a) Following the original Newtonian calculation of Laplace, show that the escape velocity from the surface of a gravitating body is equal to the speed of light if the radius of the body is equal to its gravitational radius. Discuss briefly the difference between a black hole in Newtonian theory and in General Relativity. Explain why the surface $r=r_g$ is called the event horizon. Show that this surface is null.

(b) A supermassive black hole of mass M_{bh} is surrounded by a stellar cluster. Using the result of question 2(b) from Section A, find the black hole mass, M_{crit}, such that for $M_{bh}<M_{crit}$ the tidal disruption takes place outside the black hole horizon. Express the answer in terms of the stellar parameters m_*, r_* and r_{g*}. Estimate M_{crit} if the cluster consists of solar type stars with $m_*=M_\odot$ and $r_*=R_\odot$.

(c) Assume that the luminosity of AGNs and QSOs is generated by the accretion of gas onto a supermassive black hole, where the gas comes from the tidal disruption of stars. If the luminosity is proportional to the volume V between the event horizon and the sphere of radius R_{TD}, evaluate L as a function of black hole mass and show that the maximum of L is attained at $M_{bh}=\frac{1}{\sqrt{3}}M_{crit}$.

3. (a) A binary system consists of an invisible compact object of mass M_x and a visible star of mass M. The period of the orbit is T, the angle between the normal to the plane of the orbit and the line of sight of the observer is i and the projection of the orbital velocity of the visible star on the line of sight is v. Which variables are measured directly and how? If the visible star is periodically eclipsed by the invisible object, what can you say about the orientation of the binary system?

(b) Using Newtonian theory, show that the mass function

$$f \equiv \frac{(M_x \sin i)^3}{(M_x + M)^2} = \frac{v^3 T}{2\pi G}.$$

(c) Observations of three eclipsing binaries give the following velocities and periods:

Binary number	1	2	3
Velocity in km/s	250	500	1000
Period in min	48	64	128

Assume that the invisible compact object is a black hole if its mass exceeds $3M_\odot$ and that all visible stars in the above binaries have masses between $1M_\odot$ and $10M_\odot$. By evaluating the mass function f in each case, determine which of these binaries contains, may contain or does not contain a black hole.

[End of examination paper.]