Queen Mary
University of London

M.Sc. EXAMINATION BY COURSE UNITS

ASTM003 Angular Momentum and Accretion Processes in Astrophysics

Wednesday, 29th May, 2008
18:15 – 19:45
Duration 1 hour 30 minutes

Time Allowed: 1h 30m
This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section. Marks awarded are shown next to the questions.
Calculators are NOT permitted in this examination.

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

Useful information

In this paper π and e represent the conventional mathematical constants.
G represents the gravitational constant, with $G = 6.7 \times 10^{-11}$ m^3 kg^{-1}s^{-2}.
c is the velocity of light, with $c = 3.00 \times 10^8$ m s^{-1}.
1 pc $= 3.1 \times 10^{16}$ m.
The solar mass $M_\odot = 2 \times 10^{30}$ kg.
The Stefan-Boltzmann constant $\sigma = 5.7 \times 10^{-8}$ J s^{-1} m^{-2} K^{-4}.

The isothermal sound speed c_s in a gas is related to the temperature T by

$$c_s = \sqrt{\frac{\mathcal{R}T}{\mu}} ,$$

where $\mathcal{R}$ is the gas constant and μ is the mean molecular weight.
$\mathcal{R} = 8.3$ J mol^{-1} K^{-1} and you may assume that $\mu = 2 \times 10^{-3}$ kg mol^{-1}.

The ideal gas law relates the pressure P, temperature T, and density ρ at a point in an ideal (perfect) gas by

$$P = \frac{\mathcal{R}}{\mu} \rho T.$$

The internal gravitational potential energy E_g and the moment of inertia I of a uniform sphere of mass M and radius R are

$$E_g = -\frac{3}{5}\frac{GM^2}{R} \quad \text{and} \quad I = \frac{2}{5} M R^2 .$$

You may find useful the vector identities

$$(\mathbf{A} \cdot \boldsymbol{\nabla}) f \equiv \boldsymbol{\nabla} \cdot (f\mathbf{A}) - f \boldsymbol{\nabla} \cdot \mathbf{A}$$

$$\int_V (\boldsymbol{\nabla} \cdot \mathbf{A})\, \mathrm{d}V \equiv \int_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} \qquad \text{(Divergence theorem)}$$

$$\text{and} \qquad \boldsymbol{\nabla} \cdot \mathbf{r} \equiv 3$$

for any vector $\mathbf{A}$, scalar f and position vector $\mathbf{r}$.

SECTION A

You should attempt ALL questions. Marks awarded are shown next to each question.

A1. (a) Beginning with the equation of motion in the direction z perpendicular to the plane of an accretion disc around a star of mass M, derive the equation of vertical hydrostatic equilibrium

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z} = -\frac{G\,M}{(R^2+z^2)^{3/2}}\,z\;,$$

on the assumption that the mass of the disc is negligible compared with that of the star, where ρ is the density at a point in a cylindrical coordinate system (R, ϕ, z) centred on the star, P is the gas pressure, and G is the constant of gravitation. Representing the pressure by the ideal gas law, $P = \mathcal{R}\rho T/\mu$, where $\mathcal{R}$ is the gas constant, μ is the mean molecular weight and T is the absolute temperature, solve the equation of hydrostatic equilibrium for a thin ($z \ll R$), isothermal Keplerian accretion disc to show that the density ρ at a distance z from the central plane of the disc is

$$\rho(R,\phi,z) = \rho_0(R,\phi)\,\exp(-z^2/2H^2)\;,$$

where $H(R,\phi) = \sqrt{\mathcal{R}T/\mu\Omega^2}$ and $\Omega(R)$ is the angular velocity at radius R.

[14 marks]

(b) Show that the surface density $\Sigma(R)$ at a distance R from the central star in an axisymmetric accretion disc is given by the relation $\Sigma(R) = \sqrt{2\pi}H\rho_0(R)$. The standard result

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$

may be useful. [4 marks]

(c) If the sound speed is $c_s = \sqrt{\mathcal{R}T/\mu}$, what is the relationship between c_s, H and the angular velocity Ω for the same isothermal disc? [2 marks]

[Total 20 marks for question]

A2. (a) A solid core of mass m_c has formed in a protoplanetary disc by the coagulation of planetesimals of mass m. The core has sufficient mass that its own gravitational effects on the planetesimals are significant. By considering the dynamics of the planetesimals, show that the core's cross-section for capture of these bodies is

$$A_{cap} = \pi R_c^2\left(1+\frac{2Gm_c}{R_c v^2}\right)\;,$$

where R_c is the radius of the core, v is the typical velocity of planetesimals relative to the core, and G is the constant of gravitation.
Hence show that the growth rate of the core is

$$\frac{\mathrm{d}m_c}{\mathrm{d}t} = nmv\pi R_c^2\left(1+\frac{2Gm_c}{R_c v^2}\right)\;,$$

[This question continues overleaf . . .]

where n is the number density of the planetesimals in the disc. [10 marks]
Assume that the half-thickness H_s of the planetesimal disc is related to the velocity of planetesimals by $v \simeq H_s\Omega$, where Ω is the orbital angular velocity about the central star. If ρ_s is the density of the planetesimal layer in the disc, show that when the gravitational focusing dominates over the geometrical cross-section,

$$\frac{\mathrm{d}m_c}{\mathrm{d}t} \simeq \frac{2\pi G\rho_s R_c m_c}{v^2} H_s\Omega \simeq \frac{\pi G\Sigma_s m_c^{\frac{4}{3}}}{v^2} \left(\frac{3}{4\pi\rho_{gr}}\right)^{\frac{1}{3}} \Omega ,$$

where Σ_s is the surface density of the planetesimals in the disc and ρ_{gr} is the density of material in the planetesimals. [5 marks]
Hence show that the time taken for the protoplanetary core to grow from a size $m_c(0)$ to $m_c(t_G)$ by gravitational accretion is

$$t_G \simeq \frac{3\left(m_c(0)^{-\frac{1}{3}} - m_c(t_G)^{-\frac{1}{3}}\right)}{\pi G\Sigma_s} \left(\frac{4\pi\rho_{gr}}{3}\right)^{\frac{1}{3}} \frac{v^2}{\Omega}.$$

[3 marks]

Argue that this time can be approximated by

$$t_G \simeq \frac{3m_c(0)^{-\frac{1}{3}}}{\pi G\Sigma_s} \left(\frac{4\pi\rho_{gr}}{3}\right)^{\frac{1}{3}} H_s^2\,\Omega$$

when the accreting core grows to a mass which is very much larger than its original mass. [2 marks]

[Total 20 marks for question]

A3. (a) The viscous dissipation rate per unit area for a Keplerian accretion disc is given by

$$\epsilon_D = R^2\,\nu\,\Sigma \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2 ,$$

where R is the distance from the central star, Σ is the surface density, and $\Omega(R)$ is the angular velocity about the central star. The following relation holds for a steady state Keplerian disc at a radial distance R from a non-rotating star of radius R_*:

$$\nu\,\Sigma = \frac{\dot{m}}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{\frac{1}{2}}\right] ,$$

where $\dot{m}$ is the mass accretion rate in a steady state disc. Show that at radii far from the central star, the effective temperature in a steady state disc is approximately

$$T_{eff} = \left(\frac{3GM\dot{m}}{8\pi\sigma R^3}\right)^{\frac{1}{4}} ,$$

where M is the mass of the star and σ is the Stefan-Boltzmann constant.

[6 marks]

[This question continues overleaf . . .]

(b) Estimate the temperature in an accretion disc orbiting a solar mass protostar which is accreting at a rate $\dot{m} = 10^{-8}$ $M_\odot$ yr^{-1}. You may assume that a typical radius in the disc is $R = 1.5 \times 10^{11}$ m. In which part of the electromagnetic spectrum do you expect most of the emergent energy to be radiated ? How does this compare with observation of discs around T Tauri stars ? [4 marks]

[Total 10 marks for question]

SECTION B

Each question carries 50 marks. You may attempt all questions but only marks for the best question will be counted, except for the award of a bare pass.

B1. The moment of inertia of an isolated mass of fluid is defined as

$$I = \int_V \rho\, r^2 \, \mathrm{d}V \ ,$$

where $r = |\mathbf{r}|$ is the modulus of the position vector $\mathbf{r}$ of the volume element $\mathrm{d}V$, $\rho(\mathbf{r})$ is the density, and V is a volume enclosing the fluid. The position vector is measured relative to the centre of mass of the fluid.
Show that the second derivative of the moment of inertia with respect to time t is

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 4\mathcal{K} + 2\int_V \mathbf{r}\,.\,\mathbf{f}\ \mathrm{d}V \ ,$$

where $\mathcal{K}$ is the total internal kinetic energy of the fluid and $\mathbf{f}(\mathbf{r})$ is the force per unit volume acting on the fluid. [12 marks]

The contribution to the force per unit volume from gas pressure is $\mathbf{f} = -\boldsymbol{\nabla} P$, where P is the pressure. Hence show that the contribution from gas pressure to the $2\int_V \mathbf{r}\,.\,\mathbf{f}\ \mathrm{d}V$ term is $6\int_V P\ \mathrm{d}V$, given that the pressure P on the surface S that bounds the volume V is zero. [12 marks]

Derive an expression for the contribution to $2\int_V \mathbf{r}\,.\,\mathbf{f}\ \mathrm{d}V$ from gravitational forces in terms of the total internal gravitational potential energy E_g. You may assume that $\int_V \rho(\mathbf{r})\ \mathbf{r}\,.\,\boldsymbol{\nabla}\Phi\ \mathrm{d}V = -E_g$, where $\Phi(\mathbf{r})$ is the gravitational potential. [4 marks]

Hence derive the full virial theorem for a gas cloud acting under the forces of pressure and gravity only:

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 4\mathcal{K} + 2E_g + 6\int_V P\,\mathrm{d}V$$

[3 marks]

Neglecting radiation pressure, show that the pressure contribution to $\mathrm{d}^2 I/\mathrm{d}t^2$ for an isothermal gas cloud is

$$6\int_V P\,\mathrm{d}V = \frac{6\mathcal{R}}{\mu}\, T\, M \ ,$$

where T is the temperature of the gas, M is the total mass of the cloud, μ is the mean molecular mass, and $\mathcal{R}$ is the gas constant. [4 marks]

What is the condition on $\mathrm{d}^2 I/\mathrm{d}t^2$ for the collapse of a cloud? [3 marks]

A spherical cloud of uniform density and radius R collapses under its own gravity to form a protostar. The cloud is isothermal with a temperature T, is non-magnetised, and is initially stationary. Using the virial theorem, show that a limit on the mass M of the cloud for collapse to occur is

$$M > \frac{5\mathcal{R}T}{\mu G}\, R \ ,$$

[This question continues overleaf . . .]

where $\mathcal{R}$ is the gas constant, μ is the mean molecular mass, and G is the constant of gravitation. You may assume that the internal potential energy of a uniform sphere of mass M and radius R is

$$E_g = -\frac{3}{5}\frac{GM^2}{R} .$$

[6 marks]

Use this result to estimate to within an order of magnitude the minimum mass for a cloud of radius $R = 3 \times 10^{15}$ m and temperature $T = 10$ K to collapse. Express this figure in solar masses. Can clouds of this size and temperature collapse to form protostars if they have stellar masses? [6 marks]

[Total 50 marks for question]

B2. The monochromatic radiative flux, F_ν, emitted at a particular frequency, ν, by a black–body with effective temperature T_{eff} is given by

$$F_\nu = \frac{2\pi h\nu^3}{c^2}\,\frac{1}{\exp\left(h\nu/kT_{eff}\right) - 1} ,$$

where h is the Planck constant, k is the Boltzmann constant, and c is the speed of light.

An axisymmetric accretion disc has a radial effective temperature profile given by

$$T_{eff} = \beta R^{-\alpha}$$

where α and β are positive constants. The effective temperature is T_{in} at the inner edge of the disc, and T_{out} at the outer edge. Show that the luminosity, L_ν, at a particular frequency ν, is given by

$$\tfrac{1}{2}\,L_\nu = \left(\frac{2\pi}{c}\right)^2 \frac{k^{2/\alpha}}{\alpha}\,\nu^{3-2/\alpha}\,\beta^{2/\alpha}\,h^{1-2/\alpha}\int_0^\infty \frac{x^{2/\alpha\,-\,1}}{\exp\left(x\right) - 1}\,\mathrm{d}x$$

where $x = h\nu/(kT_{eff})$. In deriving this expression, you should only consider frequencies ν such that $kT_{in} \gg h\nu \gg kT_{out}$. [32 marks]

Comment on the significance of this result for observations of accretion discs. [5 marks]

A steady state Keplerian accretion disc has

$$T_{eff} = \left[\frac{3GM\dot{m}}{8\pi\sigma R^3}\right]^{\frac{1}{4}} ,$$

where M is the mass of the central star and $\dot{m}$ is the rate of mass transfer through the disc.

Make a sketch of $\log L_\nu$ versus $\log \nu$ for such a disc, and explain the shape of the curve. Your sketch should contain the regimes with $h\nu > kT_{in}$ and $h\nu < kT_{out}$. [13 marks]

[Total 50 marks for question]

[Next question overleaf]

B3. (a) Estimate the amount of potential energy released when mass falls on to a solar mass neutron star as a fraction of the mass energy $E = mc^2$ that is contained in the same material. Assume the radius of the neutron star is 10^4 m. Give your answer accurate to within a factor of 2 if $G = 6.7 \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}$, $c = 3.0 \times 10^8$ m, and the solar mass is 2×10^{30} kg.
Is this larger or smaller than the energy available from nuclear fusion?

[4 marks]

(b) Accretion of matter on to neutron stars and black holes may occur up to a maximum rate known as the Eddington limited accretion rate. Explain briefly the physical origin of this limit. [4 marks]

(c) The radiative flux F within an optically thick gas is related to the temperature gradient $\mathrm{d}T/\mathrm{d}r$ by

$$F = -\frac{4ac}{3\kappa\rho} T^3 \frac{\mathrm{d}T}{\mathrm{d}r} ,$$

where T is the temperature, κ is a mean opacity, ρ is the density of the gas, a is the radiation constant, and c is the velocity of light. The radiation pressure within a black body of temperature T is $P_{rad} = aT^4/3$.
Derive from these the expression for the Eddington limited accretion rate, $\dot{m}_{Edd}$, of optically thick material on to a compact object of mass M,

$$\dot{m}_{Edd} = \frac{4\pi c R_c}{\kappa} ,$$

where R_c is the radius of the compact object and G is the gravitational constant. Hence show that if the radius R_c is five times the Schwarzschild radius $R_S = 2GM/c^2$,

$$\dot{m}_{Edd} = \frac{40\pi GM}{c\kappa} .$$

[20 marks]

(d) The dominant source of opacity in a hot hydrogen plasma is Thomson scattering by electrons which has $\kappa = 0.04\ \mathrm{m^2\ kg^{-1}}$. Estimate, to within an order of magnitude, the maximum mass accretion rate on to a 10^8 solar mass black hole in an active galactic nucleus in units of solar masses per year. You may use 1 year $\simeq 3 \times 10^7$ s.
Estimate also the maximum mass accretion rate onto a solar mass black hole.

[5 marks]

(e) The effective temperature T_{eff} at a distance R from the central object in an accretion disc is approximately given by

$$T_{eff} \simeq 1.2 \times 10^7\ \mathrm{K} \left(\frac{\dot{m}}{M_\odot/\mathrm{yr}}\right)^{1/4} \left(\frac{M}{M_\odot}\right)^{1/4} \left(\frac{R}{10^7\mathrm{m}}\right)^{-3/4} ,$$

where $\dot{m}$ is the mass accretion rate and M is the mass of the central object. Make an order of magnitude estimate of the effective temperatures encountered for Eddington-limited accretion on to a 10^8 solar mass black hole if a typical radius within the disc is $R \sim 10^{12}$ m. [4 marks]

[This question continues overleaf . . .]

(f) In which part of the electromagnetic spectrum would this radiation be emitted? Comment on how this compares with observations of the radiation from the inner regions of active galactic nuclei and how any discrepancies are explained.

[4 marks]

(g) Estimate the effective temperature of an accretion disc around a solar mass black hole accreting at the Eddington limit, assuming a typical radius of $R = 10^4$m. In which part of the electromagnetic spectrum will most of the radiation be emitted? In which type of astrophysical systems are stellar mass black holes usually observed ? [4 marks]

(h) What besides the Eddington limit determines the mass accretion rate through an accretion disc onto a central black hole ? [2 marks]

(i) Explain briefly the role of the boundary layer in accretion discs. Under what conditions do accretion discs around black holes or neutron stars *not* have a boundary layer ? [3 marks]

[Total 50 marks for question]

[End of examination paper]