Queen Mary
University of London

M.Sc. EXAMINATION

ASTM001 Solar System **Duration 3h**

Monday, 15 May 2006
10:00-13:00

This paper has two Sections, Section A and Section B: *<u>you should attempt both Sections</u>. Please read carefully the instructions given at the beginning of each section. You are encouraged to read through the entire paper before beginning answering the questions.*

Calculators are NOT permitted in this examination. Numerical answers, where required, may be determined approximately to within factors ~ 5.

<u>Some useful numbers and identities</u>:

- $G \approx 7 \times 10^{-11}$ m^3 Kg^{-1} s^{-2}
- $M_{\mathrm{Jupiter}} \approx 2 \times 10^{27}$ Kg
- $M_{\mathrm{Titan}} \approx 10^{23}$ Kg
- $R_{\mathrm{Jupiter}} \approx 7 \times 10^{7}$ m
- $R_{\mathrm{Titan}} \approx 3 \times 10^{6}$ m
- $2\sin^2(\theta/2) = 1 - \cos\theta$

SECTION A

Each question carries 5 marks (2.5 marks for each sub-part).
You should attempt ALL five questions.

1. Describe in one or two sentences what is meant by each of the following terms:

(a) Lamé constants

(b) Love numbers

2. Describe in one or two sentences what is meant by each of the following terms:

(a) P-wave

(b) S-wave

3. Describe in one or two sentences what is meant by each of the following terms:

(a) Lapse rate

(b) Brunt-Väisälä frequency

4. Describe in one or two sentences what is meant by each of the following terms:

(a) Rossby wave

(b) Gravity wave

5. Describe in one or two sentences what is meant by each of the following terms:

(a) Lagrange points

(b) Laplace resonance

[Next section overleaf.]

SECTION B

Each question carries 25 marks. There are 4 questions.
You may attempt all questions, but only marks for the best 3 questions will be counted.

1. Kepler's laws describe the motion of two bodies of mass m_1 and m_2 in a central force, Gm_1m_2/r^2, where G is the universal gravitational constant and r is the separation distance between the two bodies.

   (a) [6 marks] State Kepler's three laws, I, II, and III.

   (b) [10 marks] Show that the line joining the central and revolving bodies (of masses m_1 and m_2, respectively) in a Keplerian orbit sweeps out equal area in equal time and that the rate of change of the area is $h/2$, where $h = r^2\dot{\theta}$ with (r, θ) forming a polar coordinate system with m_1 at the centre.

   (c) [9 marks] Given that $h^2 = G(m_1 + m_2)\, a\,(1 - e^2)$, where a is the semi-major axis and e is the eccentricity, derive the relationship between the period of the orbit T and a.

2. A test particle approaches a planet of mass M and radius R from infinity with speed v_∞ and an impact parameter p.

   (a) [8 marks] Use the particle's energy and angular momentum with respect to the planet to derive expressions for the semi-major axis a and eccentricity e of the hyperbolic orbit followed by the test particle about M, and for the pericentre r_0.

   (b) [4 marks] Show that the eccentricity may be written $e = 1 + 2v_\infty^2/v_0^2$, where v_0 is the escape velocity at r_0.

   (c) [5 marks] Use the expression for the true anomaly corresponding to the asymptote of the hyperbola ($r \longrightarrow \infty$) to show that the overall deflection ψ of the test particle's orbit after it leaves the vicinity of the planet, is given by $\sin(\psi/2) = 1/e$.

   (d) [8 marks] Given that r_0 must be greater than R to avoid a physical collision, calculate the maximum deflection angles for (*i*) a spacecraft skimming Jupiter, with $v_\infty = 10$ km s^{-1}, and (*ii*) the Cassini orbiter skimming Saturn's large moon Titan, at $v_\infty = 5$ km s^{-1}.

3. A homogeneous, thin layer of atmosphere on a tangent plane at mid-latitude ($\sim$45°) rotates such that its local Coriolis parameter, $\vec{f} = \Omega\sqrt{2}\,\mathbf{k}$, where Ω is the rotation rate of the planet and $\mathbf{k}$ is the local vertical direction of the plane. In general, f varies

 [This question continues overleaf . . .]

spatially so that its northward gradient, $\beta \equiv df/dy = \Omega\sqrt{2}/R$, where R is the radius of the planet. The layer, with thickness $h = h(x, y, t)$, has a free surface and a constant average thickness H. The (eastward, northward) velocity, $\mathbf{v} = (u, v)$, in the layer is independent of height: $u = u(x, y, t)$ and $v = v(x, y, t)$, where (x, y) is the (eastward, northward) direction. The motion of such a layer in constant gravity g is governed by:

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} &= -g\nabla h - \vec{f} \times \mathbf{v} \\ \frac{\partial h}{\partial t} + \mathbf{v}\cdot\nabla h &= -h\nabla \cdot \mathbf{v}, \end{aligned}$$

In the layer, there is a constant eastward background flow, U, such that the total eastward velocity is: $u(x, y, t) = U + u'(x, y, t)$. There is no northward background flow – i.e., $v(x, y, t) = v'(x, y, t)$. Assume $\beta = \text{const}$ in all parts below.

(a) [9 marks] Derive the dispersion relation for a small-amplitude (linear) Rossby wave propagating in a constant background flow U with no surface variation – i.e., $h = \text{const}$, so that $(u', v') = (-\partial\psi'/\partial y, \partial\psi'/\partial x)$, where ψ' is the perturbation streamfunction.

(b) [8 marks] Now, allow the surface to vary (i.e., $h \neq \text{const}$) but with no background flow (i.e., $U = 0$). Derive the dispersion relation for a small-amplitude (linear) gravity wave supported in the layer. You may assume $f = 0$ in this case; and, $h(x, y, t) = H + h'(x, y, t)$, where $H = \text{const}$.

(c) [8 marks] Define what phase and group velocities are in terms of the wavenumber k_x. Then, derive the phase and group velocities in the $\hat{x}$-direction, v_p^x and v_g^x, for the Rossby and gravity waves found above. Explain what happens to the Rossby wave phase velocity when $U \neq 0$.

4. [25 marks; partial credit for showing steps "on the right track"] Using the standard system of units for the planar, circular restricted three-body problem, the equation defining the zero velocity curves is: $C_J = x^2 + y^2 + 2(\mu_1/r_1 + \mu_2/r_2)$, where C_J is the value of the Jacobi constant, $\mu_1 + \mu_2 = 1$, and $r_1 = \sqrt{(x + \mu_2)^2 + y^2}$ and $r_2 = \sqrt{(x - \mu_1)^2 + y^2}$ are the distances to the masses μ_1 and μ_2, respectively. Assume $\mu_2 \ll (\mu_1 + \mu_2)$. The critical zero velocity curve that passes through the L_3 equilibrium point (where $C_J \approx 3 + \mu_2$) has two branches. Use polar coordinates to show that, for small mass ratios, each of the curves crosses the unit circle at points with an angular separation of 23.9° from the secondary mass. (Hint: You may assume that the solution of $[\sin(\theta/2)]^{-1} - 2\cos\theta - 3 = 0$ is $\theta = 23.9°$.)

[End of examination paper.]