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UNIVERSITY OF LONDON
M.Sc. EXAMINATION

**ASTM001** Solar System

Friday 20th May 2005 18.15 - 21.15

*The duration of this examination is three hours.*

*Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.*

*You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best THREE questions answered will be counted.*

1 A planet of mass $m_1$ moves around a central star of mass $M_*$ located at the origin. The position vector $\mathbf{r}$ of $m_1$ satisfies the equation

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{G(M_* + m_1)\mathbf{r}}{r^3},$$

where $r = |\mathbf{r}|$.

and in cylindrical polar coordinates $(r, \theta)$ defined in the plane of the motion

$$\frac{d^2\mathbf{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\hat{\theta},$$

where $\hat{\mathbf{r}}$ and $\hat{\theta}$ are unit vectors in the $r$ and $\theta$ directions respectively. Show that $r^2(d\theta/dt) = h$, where $h$ is a constant and

$$\frac{d^2r}{dt^2} = \frac{h^2}{r^3} - \frac{G(M_* + m_1)}{r^2}. \qquad (*)$$

Deduce further that $u = 1/r$ satisfies

$$\frac{d^2u}{d\theta^2} + u = \frac{G(m_1 + M_*)}{h^2}.$$

Write down the solution that corresponds to an ellipse and relate $h$ to the semi-major axis, $a$ and the eccentricity $e$. Show further from equation $(*)$ that

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = -\frac{h^2}{2r^2} + \frac{G(m_1 + M_*)}{r} + K,$$

where $K$ is a constant.

By considering conditions at pericentre or apocentre where $dr/dt = 0$, deduce that

$$K = -\frac{G(M_* + m_1)}{2a}.$$

Hence show that the speed, $V$, at any point on the orbit is given by

$$V^2 = \left(\frac{2}{r} - \frac{1}{a}\right)G(m_1 + M_*).$$

Suppose that the earth moves in circular orbit with speed $V_E$. Assuming that $m_1$ may be neglected, deduce that the maximum speed at the earth's location of an object in a bound orbit is $\sqrt{2}V_E$.

**2** The equations of motion for the planar circular restricted three body problem can be written in Cartesian $(x, y)$ coordinates in the frame rotating with angular velocity $\Omega$ in the form

$$\frac{d^2x}{dt^2} - 2\Omega\frac{dy}{dt} = -\frac{\partial U}{\partial x},$$

$$\frac{d^2y}{dt^2} + 2\Omega\frac{dx}{dt} = -\frac{\partial U}{\partial y},$$

where

$$U = -\frac{Gm_1}{\sqrt{x^2+y^2}} - \frac{Gm_2}{\sqrt{(x-D)^2+y^2}} - \frac{1}{2}\Omega^2(x^2+y^2) + \frac{Gm_2x}{D^2}.$$

Here, the mass $m_1$ is at the origin of coordinates, $m_2$ is at $(D, 0)$ and

$$\Omega^2 = \frac{G(m_1+m_2)}{D^3}.$$

Deduce that

$$C = -\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - 2U$$

is a constant of the motion.

Equilibrium solutions are ones for which the coordinates as viewed in the rotating frame we are using do not change with time. Show from the equations of motion that these must be such that

$$\frac{Gm_1x}{(x^2+y^2)^{3/2}} + \frac{Gm_2(x-D)}{((x-D)^2+y^2)^{3/2}} + \frac{Gm_2}{D^2} - \Omega^2x = 0 \quad \text{and}$$

$$y\left(\frac{Gm_1}{(x^2+y^2)^{3/2}} + \frac{Gm_2}{((x-D)^2+y^2)^{3/2}} - \Omega^2\right) = 0.$$

Hence verify that possible equilibrium points occur with $x = D/2$, and $y = \pm\sqrt{3}D/2$. What type of triangle is formed by the lines connecting either one of these points to $m_1$ and $m_2$ together with the line joining $m_1$ and $m_2$? Describe a situation in the solar system where these solutions play an important role.

Show further that there are additional equilibrium solutions for which $y = 0$, and $x$ is a solution of the algebraic equation

$$m_1\left(\frac{D^3x}{|x|} - x^3\right) = m_2\left(x^3\left(1 + \frac{2D^2}{(D-x)^2}\right) - \frac{Dx^4}{(D-x)^2}\right).$$

Hence show that when $m_1/m_2 << 1$, which implies that $x << D$ there are two solutions such that

$$x \to \pm\left(\frac{m_2}{3m_1}\right)^{1/3} D.$$

**3** a) In a form of secular perturbation theory $i = 1, 2, 3, \dots N$ planets with masses $m_i$ obey the equations of motion

$$m_i \frac{d^2 \mathbf{r}_i}{dt^2} = -\frac{G(M_* + m_i) m_i \mathbf{r}_i}{|\mathbf{r}_i|^3} - \nabla U_i.$$

The position vectors $\mathbf{r}_i = (x_i, y_i, z_i)$ are referred to the central mass $M_*$, $t$ is time and

$$\nabla U_i = \left( \frac{\partial U_i}{\partial x_i}, \frac{\partial U_i}{\partial y_i}, \frac{\partial U_i}{\partial z_i} \right),$$

with $U_i$ being the interaction potential. For planet $i$ this is evaluated as arising from the interaction with the time averaged mass distributions of the other planets being distributed over the curves delineated by their slowly varying elliptical orbits.

For planet $i$, $U_i = U_i(x_i, y_i, z_i, t)$, is such that the effects of the planets other than planet $i$ are taken to be specified through the explicit time dependence.

Furthermore $U_i$ is presumed to be small compared to a typical orbital energy and to vary on a long time scale compared to a typical orbital period. Thus we write $|U| < \epsilon$, and $|\partial U / \partial t| < \epsilon / T$, where $\epsilon$ is a small parameter and $T$ is a long time scale.

Show that the unperturbed orbital energy of planet $i$,

$$E_i = \frac{1}{2} m_i \left( \frac{d\mathbf{r}_i}{dt} \right)^2 - \frac{G(M_* + m_i) m_i}{|\mathbf{r}_i|} \quad \text{satisfies}$$

$$\frac{dE_i}{dt} = -\frac{dU_i}{dt} + \frac{\partial U_i}{\partial t}.$$

Deduce that $E_i$ for planet $i$ is constant to within an error of order $\epsilon$ for times of order $T$.

To what well known result of secular perturbation theory does this lead?

**This question continues on the next page**

**3** b) The equations of motion for two interacting planets in secular perturbation theory are given by

$$\frac{dx_i}{dt} = \frac{1}{\sqrt{GM_* m_i a_i^{1/2}}} \frac{\partial U}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{1}{\sqrt{GM_* m_i a_i^{1/2}}} \frac{\partial U}{\partial x_i}, \quad i = 1, 2,$$

where $x_i = e_i \cos \varpi_i, y_i = e_i \sin \varpi_i$, $e_i$ being the eccentricity and $\varpi_i$ being the longitude of pericentre for planet $i$. The interaction energy is given by

$$U = -\frac{Gm_1 m_2}{8} \left( C(x_1^2 + x_2^2 + y_1^2 + y_2^2) - 2D(x_1 x_2 + y_1 y_2) \right),$$

with $C, D$ being known constants for the motion considered.

Show that the equations of motion for $\mathbf{x} = (x_1, x_2)^T$, and $\mathbf{y} = (y_1, y_2)^T$ can be written in the form

$$\frac{d\mathbf{x}}{dt} = -\mathbf{M}\mathbf{y}, \quad \frac{d\mathbf{y}}{dt} = \mathbf{M}\mathbf{x},$$

where the elements of the matrix $\mathbf{M}$ are given by $M_{11} = \kappa C$, $M_{12} = -\kappa D$, $M_{21} = -\rho D$, and $M_{22} = \rho C$ and in addition $\kappa = \frac{Gm_2}{4\sqrt{GM_* a_1^{1/2}}}$ and $\rho = \frac{Gm_1}{4\sqrt{GM_* a_2^{1/2}}}$.

Show that the general solution for $\mathbf{x}$ or $\mathbf{y}$ can be expressed as a linear combination of eigenvectors of $\mathbf{M}$ and that the eigenvalues, $\lambda$, satisfy

$$\lambda^2 - \lambda C(\kappa + \rho) - \kappa\rho(D^2 - C^2) = 0.$$

4) Two satellites with masses $m_1$ and $m_2$ ($m_1 > m_2$) orbit a central mass $M_*$ and are in a $(p+1) : p$ commensurability. The outer and more massive satellite can be assumed to be in a fixed circular orbit with constant mean motion $n_1$. The second and interior satellite with mean motion $n_2 \sim (p+1)n_1/p$ evolves according to the equation

$$\frac{dE_2}{dt} = -n_1 \frac{\partial U}{\partial \varpi_2},$$

where the resonant interaction energy

$$U = \frac{Gm_1 m_2}{a_1} f\left(a_2/a_1\right) e_2 \cos\phi,$$

with $\phi = (p+1)\lambda_1 - p\lambda_2 - \varpi_2$, where $\lambda_1$ and $\lambda_2$ are the mean longitudes of satellites 1 and 2 respectively. The quantities $(E_2, a_2, e_2, \varpi_2)$ are the orbital energy, semi-major axis, eccentricity and longitude of pericentre for satellite 2. The semi-major axis of satellite 1 is $a_1$ and the function $f(a_2/a_1)$ is a prescribed function of its argument.

Given that to an adequate approximation

$$\frac{d\phi}{dt} = (p+1)n_1 - pn_2,$$

show that $\phi$ obeys the pendulum equation

$$\frac{d^2\phi}{dt^2} = -\omega_L^2 \sin\phi$$

and give an expression for the libration frequency $\omega_L$.

Taking $\omega_L$ to be constant, explain what are the stable and unstable equilibria for the pendulum. Indicate these and draw the phase curves of the pendulum in the $(\phi, \dot{\phi})$ plane. What is the maximum value of $|\dot{\phi}|$ on the separatrix?

The inner satellite now has energy transferred to its orbit through tidal interaction with the central planet so that it now obeys the equation

$$\frac{dE_2}{dt} = -n_1 \frac{\partial U}{\partial \varpi_2} - \frac{E_2}{t_X},$$

with $t_X$ being the time scale for tidal expansion of the orbit. Show from this that an equilibrium in which tidal and resonant effects balance is possible provided the angle $\phi$ adjusts so that

$$\sin\phi = \frac{a_1 M_*}{2a_2 t_X m_1 f e_2 n_1}.$$

Where in the solar system is this type of balance important?

You may use the fact that $E_2 = -GM_* m_2/(2a_2)$.

**End of Examination**