Answer THREE questions

Mark Allocation

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

Units, Masses and Other Values

The convention: $\hbar = c = 1$ will be used throughout this paper.
The values for the following quantities may be assumed in this paper.

Meaning	Value
Masses of u,d,s,c,b,t quarks	1 MeV, 2 MeV, 0.2 GeV, 1.5 GeV, 4.5 GeV, 172 GeV
Masses of e,μ,τ leptons	0.5 MeV, 106 MeV, 1.8 GeV
Mass of all neutrinos	0
Mass of Z boson ($M_{\rm Z}$)	91 GeV
Mass of W boson ($M_{\rm W}$)	80 GeV
Fermi Weak Decay Constant ($G_{\rm F}$)	1.11×10^{-5} GeV^{-2}

CKM Matrix

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} = \begin{pmatrix} 0.974 & 0.227 & 0.004 \\ 0.227 & 0.973 & 0.042 \\ 0.008 & 0.042 & 0.999 \end{pmatrix}$$

where V_{ij} is the factor for interactions involving quarks i and j.

Dirac Matrices

The Dirac γ matrices satisfy $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ (for $\mu, \nu =$ 0,1,2,3) are defined as:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^{i=1,2,3} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

And the Pauli spin matrices, σ_i, are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which satisfy: $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{c}) = \vec{a}\cdot\vec{c} + i\vec{\sigma}\cdot(\vec{a}\times\vec{c})$ for 3 component vectors $\vec{a}$, $\vec{c}$.

[Part marks]

1. (a) Draw the highest rate, lowest order Feynman diagram for an electron scattering from an up-quark which produces an electron-neutrino ν_e in the final state.
Write down the factors at each interaction vertex in terms of the coupling constant, Dirac γ matrices and elements of the CKM matrix. **[7]**

(b) Neglecting the mass of the electron and the ν_e show that if q is the difference in 4-momentum between the electron and the ν_e that:

$$Q^2 = -q^2 = 4E_e E_\nu \sin^2 \tfrac{\theta}{2},$$

where E_e and E_ν are the energy of the electron and ν_e respectively and the ν_e emerges at an angle of θ with respect to the original electron direction. **[5]**

(c) For $Q^2 = 10$ GeV^2, neglecting the coupling constants, estimate the ratio of up-quark electron interactions producing a ν_e in the final-state to up-quark electron interactions producing an electron in the final-state. **[3]**

(d) Estimate what fraction of positron proton collisions at $\sqrt{s} = 300$ GeV, $Q^2 = 10$ GeV^2 and $x = 0.33$ will produce a charm quark and a $\overline{\nu}_e$ in the final state, where x is the fraction of the proton's 4-momentum carried by the struck quark. You should consider only up and down quarks and their anti-quarks in the initial proton. **[5]**

[Part marks]

2. The Lagrangian density of QED is:

$$\mathcal{L}_{\mathrm{QED}} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi + e\bar{\psi}\gamma_{\mu}A^{\mu}\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

where $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$.

(a) If A^{μ} undergoes a gauge transformation of the form $A^{\mu} \rightarrow A^{\mu} + \partial^{\mu}\chi$ show that $F^{\mu\nu}$ is unchanged. **[3]**

(b) Show that $\mathcal{L}_{\mathrm{QED}}$ is not invariant when ψ undergoes a local phase transformation of the form $\psi \rightarrow \psi e^{ie\theta}$ where θ depends on position. **[5]**

(c) What values does χ have to take in order to restore the invariance of $\mathcal{L}_{\mathrm{QED}}$ under a local phase transformation? **[2]**

Consdering the production of a Higgs boson of mass 120 GeV from a pp collision at the LHC.

(d) Draw the dominant Feynman diagram producing such a Higgs boson in association with a $t\bar{t}$ pair. **[3]**

(e) Draw the Feynman diagram illustrating the dominant decay modes of the Higgs boson and the t-quarks and $\bar{t}$-quarks. Considering only these decays, how many b-quarks and $\bar{b}$-quarks are expected in the final state? **[3]**

(f) Experimentally how would one detect a b-quark and determine its lifetime in its own rest-frame? **[4]**

[Part marks]

3. (a) The Lagrangian density of QCD is:

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi - g_s\bar{\psi}\gamma^{\mu}\mathbf{G}_{\mu}\psi - \tfrac{1}{4}\sum_{a=1}^{8} G^{a}_{\mu\nu}G^{a\mu\nu}.$$

Explain using Feynman diagrams what the three terms in the expression for $\mathcal{L}_{\mathrm{QCD}}$ correspond to physically. **[4]**

(b) Give three pieces of experimental evidence that support the existence of the QCD property of colour. **[3]**

(c) Explain what is meant by the procedure of renormalisation and the problem it solves. **[2]**

(d) Draw the two simplest Feynman diagrams that are responsible for screening and anti-screening the quark colour charge and hence explain the differing momentum-transfer dependence of the EM and QCD coupling constants. **[3]**

(e) For e^+e^- collisions at $\sqrt{s} = 5$ GeV, estimate the ratio of the rates at which interactions produce hadrons and $\mu^+\mu^-$. **[4]**

(f) How would one expect the ratio of the rates at which interactions produce hadrons and $\mu^+\mu^-$ to change if the e^+e^- collisions were at $\sqrt{s} = 90$ GeV? **[4]**

[Part marks]

4. (a) Draw the Feynman diagrams showing the dominant leptonic decay mode of the μ^- and the τ^-.
Assuming $m_\tau \gg m_\mu \gg m_e$, estimate the ratio of the rate of the two decay modes. **[4]**

(b) Draw the Feynman diagram producing a τ^- from an interaction of a ν_τ and an electron. **[2]**

(c) What is the minimum energy of the ν_τ incident on a stationary electron required to produce a τ^- in the limit that the outgoing neutrino energy, E_ν, is zero? **[4]**

(d) Consider the decay $\pi^- \rightarrow \mu^- \overline{\nu}_\mu$, where the μ^- and $\overline{\nu}_\mu$ both emerge at an angle of 30^0 with respect to the original π^- direction. If the π^- has an energy defined by a Lorentz γ factor of $\sqrt{3}$, show that the minimum energy of the $\overline{\nu}_\mu$, E_ν, is given by:

$$E_\nu = \frac{1}{\sqrt{3}} \left(m_\pi - m_\mu \right),$$

where m_π and m_μ are the rest-masses of the π^- and μ^- respectively. **[7]**

(e) Draw a diagram illustrating the spin and momentum vectors of the μ^- and $\overline{\nu}_\mu$ from the decay of a π^- at rest. Explain briefly why this decay mode is preferred over the decay $\pi^- \rightarrow e^- \overline{\nu}_e$. **[3]**

[Part marks]

5. (a) Show that $(\gamma^\mu)^\dagger = g_{\mu\nu}\gamma^\nu$ and hence that $(\gamma^\mu)^\dagger \gamma^0 = \gamma^0\gamma^\mu$. **[5]**

(b) By taking the Hermitian conjugate of the Dirac equation: $(i\gamma^\mu\partial_\mu - m)\, u = 0$ and multiplying by γ^0 and using the result of part (a), show that the adjoint Dirac equation is:

$$\overline{u}\,(i\gamma^\mu\partial_\mu + m) = 0.$$

[5]

(c) Consider the solution to the Dirac equation, u_A, defined as:

$$u_A = \sqrt{|E| + m}\begin{pmatrix} \chi_A \\ \left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)\chi_A \end{pmatrix} \quad \text{where } \chi_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and show that $u_A^\dagger u_A = 2E$. **[6]**

(d) By considering u_A in the massless limit and the helicity operator, $\hat{h}$, defined as:

$$\hat{h} = \frac{1}{2}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix},$$

show that

$$\hat{h}u_A = \tfrac{1}{2}\gamma^5 u_A$$

and briefly explain the significance of this result. **[4]**

END OF PAPER