Answer ALL questions from SECTION A and TWO questions from SECTION B

The numbers in square brackets in the right hand margin indicate the provisional allocation of maximum marks per sub–section of a question.

You may assume the following:

Equation of Hydrostatic Equilibrium:

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho(r)$$

Equation of Mass Continuity:

$$\frac{dm}{dr} = 4\pi r^2 \rho(r)$$

Exponential Integral:

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} \mathrm{d}t$$

The Equation of State for an Ideal gas:

$$P = \frac{\rho k T}{\mu m_H}$$

Radiation Pressure:

$$P_{\rm rad} = \frac{a}{3} T^4$$

Gravitation constant $G = 6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

Solar radius $R_\odot = 6.96 \times 10^8$ m

Solar mass $M_\odot = 1.99 \times 10^{30}$ kg

SECTION A

1. The equation of radiative transfer is given by (using standard notation)

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu.$$

By integrating over solid angle and assuming that the source function S_ν is isotropic, derive the first moment of the transfer equation as given by (using standard notation)

$$\frac{dH_\nu}{d\tau_\nu} = J_\nu - S_\nu$$

[7]

2. By combining the equations of hydrostatic equilibrium and mass continuity, show that the lower limit for the central pressure P_c of a star in hydrostatic equilibrium is given by

$$P_c > \frac{GM_*^2}{8\pi R_*^4}$$

where M_* and R_* represent the total stellar mass and radius respectively. **[7]**

3. Define what is meant by the *monochromatic opacity* χ_ν. **[2]**

Describe the two main sources of stellar opacity deep inside a stellar interior. **[2]**

What is the main source of opacity in the atmospheres of (i) A stars; (ii) the Sun; and (iii) OB stars? **[3]**

4. Describe the main differences between *Upper* and *Lower* main sequence stars. **[7]**

5. The dynamical timescale is given by:

$$\tau_{\rm dyn} = \left(\frac{R^3}{2GM}\right)^{1/2}.$$

Explain the meaning of this timescale in stellar evolution and provide an example of an evolutionary phase that operates on this timescale. **[3]**

Calculate the dynamical timescale for the Sun. **[3]**

6. Describe in general terms how mass loss affects the evolution of stars, quoting specific examples of stellar evolutionary phases where appropriate. **[6]**

SECTION B

7. The equation of radiative transfer for a plane-parallel stellar atmosphere is given by (using standard notation):

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu \;-\; S_\nu(\tau_\nu).$$

Derive the formal solution to this equation for I_ν as given by **[13]**

$$I(\tau_\nu, \mu, \nu) = \int_{\tau_\nu}^{\infty} S_\nu(t) \exp[\frac{-(t-\tau_\nu)}{\mu}] \,\frac{dt}{\mu} - \int_0^{\tau_\nu} S_\nu(t) \exp[\frac{-(t-\tau_\nu)}{\mu}] \,\frac{dt}{\mu}.$$

Show that this leads to the Schwarzschild-Milne relation:

$$J_\nu(\tau_\nu) \quad = \quad \frac{1}{2}\int_0^{\infty} S_\nu(t)\; E_1\,(\; |t-\tau_\nu| \;)\, \mathrm{d}t,$$

where E_1 is the first exponential integral. **[9]**

If the envelope of a star obeys (i) an opacity law of the form $\kappa = \kappa_0 P/T^4$ and (ii) the equation of radiative transport

$$L(r) = -\frac{16\pi ac}{3\kappa\rho}\; r^2 T^3 \frac{\mathrm{d}T}{\mathrm{d}r},$$

show that the variation of pressure P with temperature T obeys the approximate relation **[8]**

$$P = \left(\frac{4\pi acGM}{3\kappa_0 L}\right)^{1/2} \; T^4.$$

You may assume that the star is in hydrostatic equilibrium and that the mass M and luminosity L are constant in the envelope.

8. A bubble of gas is in pressure equilibrium with its surroundings. The temperature of the bubble is perturbed such that its new temperature is slightly higher than that of its surroundings. By considering the buoyancy force, explain under what conditions convection will occur, and show that for stability *against* convection,

$$\left|\frac{dT}{dr}\right|_{\rm rad} < \left|\frac{dT}{dr}\right|_{\rm ad}$$

where $\left|\frac{dT}{dr}\right|_{\rm rad}$ is the radiative temperature gradient, applicable to the surroundings, and $\left|\frac{dT}{dr}\right|_{\rm ad}$ is the temperature gradient experienced by the bubble as it rises adiabatically. You may assume that the bubble is in pressure equilibrium with its surroundings. **[12]**

Hence show that the *Schwarzschild criterion* for stability *against* convection for an adiabatic gas of temperature T and pressure P, is given by **[9]**

$$\left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right|_{\rm rad} < \left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right|_{\rm ad}$$

and that the adiabatic gradient is given by

$$\left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right|_{\rm ad} = \frac{dT}{dP}\frac{P}{T} = \left(\frac{\gamma - 1}{\gamma}\right).$$

The material in a star obeying hydrostatic equilibrium has a ratio of specific heats $\gamma = 5/3$, and also satisfies the equation of state for an ideal gas. The star is *convectively unstable* in such a way to *just* satisfy the criterion for convective stability.

Show that the radiative temperature gradient within the star is given by:

$$\left|\frac{dT}{dr}\right|_{\rm rad} = \frac{2\,\mu m_{\rm H} G M}{5 k r^2}$$

where $\mu m_{\rm H}$ is the mean molecular weight of the stellar material, k is the Boltzmann constant, G is the Gravitational constant, r is the stellar radius, and M is the stellar mass. **[9]**

9. Give the expression for the polytropic equation of state, explaining each term. Give two examples of stars that behave as polytropes, and write down their polytropic equations of state. **[7]**

A star of uniform composition μ and mean density ρ has a total pressure P which is a mixture of gas pressure $P_{\rm gas}$ and radiation pressure $P_{\rm rad}$. If $P_{\rm gas}/P = \beta$ (where β is a constant), show that the relationship between P and ρ is that of a polytrope. **[8]**

Given that such a star can be represented by a polytrope with an index $n = 3$, Eddington showed that its mass is given by

$$\frac{M}{M_\odot} = \frac{18.0}{\mu^2}\frac{\sqrt{(1-\beta)}}{\beta^2}.$$

Determine the stellar mass at which the gas and radiation pressures are equal. You may assume $\mu = 0.6$. **[5]**

The mass-radius relationship for a polytrope of index n with $1 \le n \le 3$ is given by

$$R^{3-n} \propto \frac{1}{M^{n-1}}.$$

Explain why consideration of this equation for non-relativistic and relativistic degenerate electron gases, leads to an upper limit to the mass of a white dwarf. **[10]**

10. An intermediate mass ($2\,M_\odot \le$M$\le 8\,$M$_\odot$) star is thought to undergo three dredge-up episodes during its evolution. For each dredge-up, describe the evolutionary state of the star, its structure and the products that are brought to the surface. **[12]**

Describe the mechanism by which thermal pulsations operate in the Thermally Pulsating Asymptotic Giant Branch (TP-AGB) phase of stellar evolution. **[12]**

In the AGB phase, the luminosity L of the star is strictly related to the mass of the core by the "Paczynski relation".

Why is L uniquely determined by the core mass (and not the total stellar mass) during this evolutionary phase?
Hence explain why an AGB star evolves vertically upwards in the Hertsprung-Russell diagram. **[6]**