Answer THREE questions

The numbers in square brackets in the right hand margin indicate the provisional allocation of maximum marks per sub–section of a question.

You may assume the following:

Gravitation constant $G = 6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

Mean solar density $\bar{\rho}_\odot = 1400\ \mathrm{kg\,m^{-3}}$

Solar radius $R_\odot = 6.96 \times 10^8$ m

Mean absorption coefficient for solar interior $\kappa = 0.04\ \mathrm{m^2\,kg^{-1}}$

1. Derive the equation of hydrostatic equilibrium for a spherically symmetric star. **[3]**

   By combining the equations of hydrostatic equilibrium and mass continuity, show that the lower limit for the central pressure P_c of a star in hydrostatic equilibrium is given by

   $$P_c > \frac{GM^2}{8\pi R^4}$$

   (Hint: you should replace r by the stellar radius R where appropriate.) **[5]**

   Estimate the mean free path of a photon at an "average" point inside the sun. Hence explain, by reference to the sun, why radiative transport in stellar interiors can be treated as a diffusive process. **[4]**

   The diffusive flux $\vec{\jmath}$ of particles (per unit area and time) between places of different particle density n is given by

   $$\vec{\jmath} = -D\nabla n$$

   where the coefficient of diffusion $D = \frac{1}{3}v\ell_{\mathrm{p}}$ is determined by the average values of mean velocity v and mean free path ℓ_{p} of the particles.

   Show that for the case of a stellar interior, where photons are the transporting particles with a radiation energy density $U = aT^4$, and assuming spherical symmetry, that the equation of radiative transport is given by

   $$\frac{\partial T}{\partial r} = -\frac{3}{16\pi ac}\,\frac{\kappa\rho}{r^2T^3}\,L(r)$$

   where a is the radiation-density constant, κ is the absorption coefficient, ρ is the density and $L(r)$ is the luminosity. **[8]**

2. Give expressions for the first three moments of a radiation field of specific intensity I_ν in the plane-parallel approximation. **[4]**

Explain the concept of radiative equilibrium and for a plane-parallel geometry, write down an expression for the radiative flux when this approximation applies (the "first condition of radiative equilibrium"). **[2]**

Given that the equation of radiative transfer for a plane-parallel atmosphere can be expressed as

$$\mu \, \frac{\mathrm{d}I_\nu}{\mathrm{d}x} = \chi_\nu \; I_\nu \; - \; \chi_\nu \; S_\nu,$$

derive the second and third conditions for radiative equilibrium: **[6]**

$$\int_0^\infty \chi_\nu \; J_\nu \; \mathrm{d}\nu = \int_0^\infty \chi_\nu \; S_\nu \; \mathrm{d}\nu \qquad \text{and} \qquad \frac{\mathrm{d}K_\nu}{\mathrm{d}\tau_\nu} = \frac{\mathcal{F}_\nu}{4\pi}.$$

Assuming that the opacity χ_ν is grey and that I_ν can be represented by a constant inward and outward term (the Eddington approximation), derive the relationships between $J, \mathcal{F}$ and K. Hence show that the source function is given by

$$S(\tau) = \frac{3}{4\pi} \; (\tau + \frac{2}{3}) \; F_0$$

where $F_0 = \sigma T_{\mathrm{eff}}^4$ is the flux constant. **[8]**

3. List the physical processes that give rise to the continuous opacity in stars. **[3]**

For each source of continuous opacity, describe the conditions needed for it to be significant in a stellar atmosphere and indicate the required effective temperature range. **[6]**

For an adiabatic gas of temperature T and pressure P, show that the *Schwarzschild criterion* for stability *against* convection is given by **[6]**

$$\left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right|_{\mathrm{rad}} < \left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right|_{\mathrm{ad}}$$

and that the adiabatic gradient is given by

$$\left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right|_{\mathrm{ad}} = \left(\frac{\gamma - 1}{\gamma}\right).$$

Explain why stars like the sun have outer convective zones and why massive stars have convective cores. **[5]**

4. State what is meant by a *polytropic process* and write down the polytropic equation of state. **[2]**

Derive the Lane-Emden equation for a polytrope of index n as given by

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{dw}{dz}\right) + w^n = 0$$

where w and z are dimensionless variables defined by $\rho = \rho_c\, w^n$ $(0 \le w \le 1)$; and $z = r/A$ where **[8]**

$$A^2 = \left[\frac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}}\right]$$

Determine the central pressure of *Capella* assuming it can be represented by a polytropic model of index $n = 3$ with a mass of 8.3×10^{30} kg, and a radius of 9.55×10^9 m. [You may assume $z_3 = 6.897$ and $\rho_c/\overline{\rho} = 54.18$.] **[5]**

The mass-radius relationship for a polytrope of index n with $1 \le n \le 3$ is given by

$$R^{3-n} \propto \frac{1}{M^{n-1}}.$$

Explain why consideration of this equation for non-relativistic and relativistic degenerate electron gases, leads to an upper limit to the mass of a white dwarf. **[5]**

5. Define the three characteristic timescales of stellar evolution and give an example of an evolutionary phase that operates on each timescale. **[6]**

Describe the main characteristics of *Upper* and *Lower* main sequence stars. **[5]**

An intermediate mass $(2\,\mathrm{M}_\odot \le M \le 8\,\mathrm{M}_\odot)$ star is believed to experience three dredge-up episodes during its evolution. For each dredge-up, describe the evolutionary state of the star, its structure and the products that are brought to the surface. **[9]**