Answer THREE questions.

The numbers in the square brackets in the right-hand margin indicate the provisional allocation of marks per subsection of a question.

1. A particle has angular momentum operator $\mathbf{J}$. Its angular momentum quantum number $j = 1$.

   What are the eigenvalues of (a) J^2, (b) its z-component J_z? **[2]**

   Write down the matrices representing J^2 and J_z in the basis formed by their normalised joint eigenvectors $\mid j,\ m >$. **[2]**

   By solving the secular equation show that these eigenvectors are given by

   $$\mid 1,\ 1 >= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ; \quad \mid 1,\ 0 >= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ; \quad \mid 1,\ -1 >= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$ **[2]**

   Verify that they are orthogonal and also complete, i.e.

   $$\sum_{m=-1}^{m=1} \mid 1,\ m >< 1,\ m \mid = I$$

   where I is the unit matrix of dimension 3. **[4]**

   The operators J_+ and J_- are defined by

   $$J_+ = J_x + iJ_y; \qquad J_- = J_x - iJ_y.$$

   Assuming without proof that

   $$J_\pm \mid j,\ m >= \hbar\sqrt{j(j+1) - m(m \pm 1)} \mid j,\ m \pm 1 >$$

   find the matrix of J_x in the same basis. **[4]**

   Show that the eigenvalues of J_x are the same as those of J_z.

   The corresponding normalised eigenvectors are

   $$\frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} ; \quad \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} ; \quad \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} .$$

Find the values obtained and their probabilities when the system is initially in the state

$$| \psi >= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$ **[6]**

2. The Hamiltonian operator H for a one-dimensional harmonic oscillator of mass m and angular frequency ω is

$$H = \hbar\omega(a_+ a_- + 1/2),$$

where a_+ and a_- are the creation and annihilation operators defined by

$$a_+ = \frac{1}{\sqrt{2m\hbar\omega}}(p + im\omega x),$$

$$a_- = a_+^\dagger,$$

and x and p are position and momentum operators satisfying $[x, p] = i\hbar$. From these definitions show that

$$[a_-, a_+] = 1$$ **[2]**

and

$$[H, a_+] = a_+\hbar\omega$$
$$[H, a_-] = -a_-\hbar\omega.$$ **[2]**

Using the notation $H \mid n >= E_n \mid n >$, show that

$$Ha_+ \mid n >= (E_n + \hbar\omega)a_+ \mid n >; \qquad Ha_- \mid n >= (E_n - \hbar\omega)a_- \mid n > .$$ **[2]**

Show that the lowest eigenvalue of H, assuming one exists, $E_0 = \frac{1}{2}\hbar\omega$ and that

$$E_n = (n + \frac{1}{2})\hbar\omega.$$

and further that the eigenvector corresponding to eigenvalue E_n is

$$\mid n >= A_n(a_+)^n \mid 0 > .$$ **[5]**

where n is an integer $(n \geq 1)$ and A_n is a normalisation constant.

Show that, if C_0, C_1 and C_n are also normalisation constants, the oscillator wave functions in the x-representation are

$$\psi_0(x) = C_0 e^{-m\omega x^2/2\hbar}$$ **[3]**

$$\psi_1(x) = C_1 x e^{-m\omega x^2/2\hbar}$$ **[3]**

$$\psi_n(x) = C_n \left(-i\hbar \frac{d}{dx} + im\omega x \right)^n e^{-m\omega x^2/2\hbar}.$$ **[3]**

3. The Hamiltonian matrix of a quantum system is given by

$$\mathbf{H} = E_0 \begin{pmatrix} 1+\lambda & 0 & 0 \\ 0 & 8 & -2\lambda \\ 0 & -2\lambda & 3 \end{pmatrix}$$

where E_0 is a real constant and λ is a real, positive constant with $\lambda << 1$.

(a) Decompose this Hamiltonian into unperturbed and perturbed parts in the form

$$\mathbf{H} = \mathbf{H_0} + \lambda \mathbf{V}$$ **[1]**

(b) Find the eigenvalues of $\mathbf{H_0}$. **[1]**

(c) By diagonalising $\mathbf{H}$ find its exact eigenvalues. **[6]**

(d) Find approximate eigenvalues of $\mathbf{H}$ as given by second order perturbation theory. **[7]**

(e) Show that the exact and approximate eigenvalues are the same to order λ^2. **[5]**

You may assume the second-order perturbation theory formula

$$W_n = E_n + < u_n \mid \lambda V \mid u_n > + \sum_{m \neq n} \frac{|< u_m \mid \lambda V \mid u_n >|^2}{E_n - E_m}.$$

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4. By means of an illustrative example, explain the statement that Quantum Mechanics is a probabilistic as opposed to a deterministic theory. **[2]**

What are meant in Quantum Mechanics by:

Hidden variable theories? **[2]**

Entanglement? **[2]**

Non-locality ? **[2]**

Describe how the criteria proposed by Einstein, Podolsky and Rosen as a basis of an acceptable quantum theory were put to experimental test by Aspect and co-workers. What were the results of the experiment and what conclusions could be drawn from these results? **[6]**

Discuss how quantum mechanical principles can be applied to

EITHER (a) Quantum Cryptography OR (b) Quantum Teleportation. **[6]**

5. The Hamiltonian operator H describing a quantum mechanical system in spherical polar co-ordinates has a lowest energy eigenvalue E_0. Show, for any normalisable function $F(\mathbf{r})$ that satisfies the boundary conditions appropriate to a bound state, that the expectation value $E(F)$ of H satisfies

$$E(F) = \frac{\int F(\mathbf{r})^* H F(\mathbf{r}) d\mathbf{r}}{\int F(\mathbf{r})^* F(\mathbf{r}) d\mathbf{r}} \geq E_0.$$

Explain how this expression can be used to find an approximation to the ground state energy which is an upper limit on its value. **[7]**

Use a trial function of the form

$$F(\mathbf{r}, \alpha) = e^{-\alpha r/2}$$

where α is a variational parameter, to investigate the properties of a particle of mass m in a central potential of the form

$$V(r) = -V_0 \frac{e^{-r}}{r}$$

where V_0 is a positive constant.

Show that an upper bound on the ground state energy may be written

$$E(\alpha) = \frac{\hbar^2 \alpha^2}{8m} - \frac{V_0 \alpha^3}{2(\alpha+1)^2}$$ **[5]**

where α is a solution of the equation

$$\frac{\hbar^2 (1+\alpha)^3}{2m} = V_0 \alpha (3+\alpha)$$ **[4]**

Determine the smallest value of V_0 such that the variational method predicts that there is just one bound state. **[4]**

You may assume the following:

$$\int_0^\infty e^{-ax} x^n dx = \frac{n!}{a^{n+1}}$$

where n is an integer ≥ 0 and $a > 0$.

In spherical polar co-ordinates, $d\mathbf{r} = r^2 \sin\theta dr d\theta d\phi$, and

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

END OF PAPER