UNIVERSITY COLLEGE LONDON

PHYSICS 2B72 MATHEMATICAL METHODS FOR PHYSICS 2001

All questions may be attempted.
Full marks will be given for correct answers to about four questions.
The numbers in square brackets in the right-hand margin indicate the provisional allocation of marks per sub-section of a question.

1. If $\mathbf{A}$ and $\mathbf{B}$ are both matrices of dimension $n \times n$, prove that

$$(\mathbf{BA})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}$$

and hence show that the matrix $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is symmetric. [4]

If $\mathbf{A}$ is also non-singular prove that

$$\mathbf{A}^{-1} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}. \qquad (a)$$ [2]

The matrix $\mathbf{A}$ is defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & -5 \\ -4 & 1 & 11 \\ 7 & 2 & 7 \end{pmatrix}.$$

Find the inverse matrix $\mathbf{A}^{-1}$. [7]

Verify this result by also using the relation (a) above to evaluate $\mathbf{A}^{-1}$. [7]

2. Define what is meant by a Hermitian matrix and a unitary matrix. [2]

Prove that the eigenvalues of a Hermitian matrix are real and that, if all the eigenvalues are distinct, the eigenvectors are mutually orthogonal. [8]

The matrix $\mathbf{H}$ is defined by

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2-\mathrm{i} \\ 0 & 2+\mathrm{i} & 0 \end{pmatrix},$$

where $\mathrm{i} = \sqrt{-1}$. Show that $\mathbf{H}$ is Hermitian and that $\mathbf{H}^{\dagger}\mathbf{H}$ is diagonal. [3]

Find the eigenvalues of $\mathbf{H}$ and determine the corresponding normalised eigenvectors. [6]

Verify that these eigenvalues are consistent with the result obtained for the diagonal matrix $\mathbf{H}^{\dagger}\mathbf{H}$. [1]

3. The function $V(r,\theta,\phi)$ is a scalar function of position and satisfies Laplace's equation in spherical polar coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0\,.$$

Use the method of separation of variables to show that a solution of the type

$$V(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)$$

may be obtained and prove that the radial function $R(r)$ satisfies an equation of the form

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-l(l+1)R=0\,,$$

where l is a real constant. [12]

By using the trial function $R(r)=r^k$ find two solutions of this equation. [3]

Verify that, for each of the two values of k for $l=2$, the function

$$V(r,\theta,\phi)=r^k(3\cos^2\theta-1)$$

is a solution of Laplace's equation. [5]

4. A generating function, $G(x,h)$, for the Legendre polynomials, $P_l(x)$, is defined by

$$G(x,h)\equiv(1-2xh+h^2)^{-\frac{1}{2}}=\sum_{l=0}^{\infty}P_l(x)h^l;\qquad |h|<1,\quad |x|\leq 1.$$

By differentiating $G(x,h)$ with respect to h or x, obtain the recurrence relations

$$(l+1)P_{l+1}(x)-(2l+1)xP_l(x)+lP_{l-1}(x)=0\qquad (a)$$ [7]

and

$$P_l(x)=P'_{l+1}(x)-2xP'_l(x)+P'_{l-1}(x)\,,\qquad (b)$$ [6]

valid for $l\geq 1$.

By differentiating result (a) and combining it with result (b) show that

$$(2l+1)P_l(x)=P'_{l+1}(x)-P'_{l-1}(x)\,.$$ [3]

Obtain an expression for the integral of $P_l(x')$ between the limits $x'=-1$ and $x'=x$. [2]

Given that $P_1(x)=x$, $P_2(x)=(3x^2-1)/2$ and $P_3(x)=(5x^3-3x)/2$, verify this expression for $l=2$. [2]

5.. The function $y(x)$ satisfies the second-order differential equation

$$\frac{d^2y}{dx^2} - \frac{\nu(\nu+1)}{x^2}y + \alpha^2 y = 0,$$

where α and ν are real constants. Show that this equation has two independent solutions, $y_1(x)$ and $y_2(x)$, of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+k}$$

with $k = \nu + 1$ and $k = -\nu$. [5]

Derive the recurrence relation

$$\frac{a_n}{a_{n-2}} = -\alpha^2[(n+k+\nu)(n+k-\nu-1)]^{-1}.$$ [5]

Hence or otherwise show that for $\nu = 1$, the function

$$y(x) = (\alpha x)^{-1}\sin(\alpha x) - \cos(\alpha x)$$ [7]

is a solution of the differential equation.

What functions $y(x)$ are independent solutions of the differential equation for $\nu = 0$? [3]

6. The function $f(x)$ is periodic with period 2π and is continuous within the interval $-\pi < x < \pi$. If $f(x)$ has a Fourier expansion of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

then prove that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x\,; \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x\,.$$ [8]

If $f(x) = x$ for $-\pi \leq x \leq \pi$, evaluate the coefficients a_n and b_n. [6]

Hence, by selecting an appropriate value for x, show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} = \frac{\pi}{4}.$$ [6]

7. If ϕ and $\mathbf{A}$ are scalar and vector fields, give expressions for

$$\nabla\phi, \quad \nabla.\mathbf{A}, \quad \nabla \times \mathbf{A}, \quad \nabla^2\phi$$ [2]

in Cartesian coordinates.

Hence prove that

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A}\,.$$ [6]

State Stokes' theorem. [3]

Hence prove that for any vector field $\mathbf{A}$

$$\int_S [\nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A}].\mathbf{dS} = \int_C \nabla \times \mathbf{A}.\mathbf{ds}\,, \qquad (a)$$

where S is a surface bounded by a curve C. The element of surface $\mathbf{dS} = \hat{\mathbf{n}}dS$ where $\hat{\mathbf{n}}$ is a unit vector along the outward normal to S and $\mathbf{ds}$ is a line element along contour C whose direction obeys a righthand screw rule w.r.t. $\hat{\mathbf{n}}$. [3]

A rectangular block is specified by the conditions

$$0 \le x \le 1 \quad ; \quad 0 \le y \le 1 \quad ; \quad 0 \le z \le 2\,.$$

If the surface S includes all surfaces of the block except the one lying in the $x-y$ plane, verify result (a) for the case

$$\mathbf{A} = z^2x^2\,\hat{\mathrm{k}}\,.$$ [6]

END OF PAPER