UNIVERSITY COLLEGE LONDON

PHYSICS 2B72 MATHEMATICAL METHODS FOR PHYSICS 2000

All questions may be attempted.
Full marks will be given for correct answers to about four questions.
The numbers in square brackets in the right-hand margin indicate the provisional allocation of marks per sub-section of a question.

1. a) The vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are defined by

$$(i) \qquad \mathbf{u} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}; \quad \mathbf{v} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}; \quad \mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

and

$$(ii) \qquad \mathbf{u} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}; \quad \mathbf{v} = \begin{pmatrix} -2 \\ 6 \\ 7 \end{pmatrix}; \quad \mathbf{w} = \begin{pmatrix} -1 \\ 10 \\ 12 \end{pmatrix}.$$

In each case, determine whether $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent or not. If they are linearly dependent, determine the relation between them. [5]

b) If $\mathbf{A}$ is a symmetric non-singular matrix of dimension $n \times n$, show that $\mathbf{A}^{-1}$ is also symmetric. [3]

The quantities x_1, x_2 and x_3 satisfy the equations

$$x_1 + x_2 + \mathrm{i}x_3 = 1$$

$$x_1 + \mathrm{i}x_2 - x_3 = 1$$

$$\mathrm{i}x_1 - x_2 + \mathrm{i}x_3 = -2,$$

where $\mathrm{i} = \sqrt{-1}$. Write these equations in the form $\mathbf{Ax} = \mathbf{b}$, and determine the matrix $\mathbf{A}^{-1}$. [8]

Hence show that the solution of the simultaneous equations is given by

$$x_1 = 1, \qquad x_2 = 1 + \frac{1}{2}\mathrm{i}, \qquad x_3 = -\frac{1}{2} + \mathrm{i}, \qquad [2]$$

and obtain the normalised vector that corresponds to $\mathbf{x}$. [2]

2. If **A**, **B** and **T** are square matrices of dimension $n \times n$ and

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T},$$

where **T** is non-singular, show that the eigenvalues of **A** and **B** are identical. [5]

Two additional $n \times n$ matrices **C** and **D** are such that

$$\mathbf{D} = \mathbf{T}^{-1}\mathbf{C}\mathbf{T}$$

and **B** and **D** commute. Prove that **A** and **C** also commute. [3]

If **A** and **C** are defined by

$$\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 4 & 1 \end{pmatrix}; \qquad \mathbf{C} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix},$$

find the eigenvalues and eigenvectors of **A**. [6]

Hence, given that **B** is diagonal, obtain **T** and show that **D** is also diagonal. [4]

Verify that **A** and **C** commute. [2]

3. a) The function $U(x,t)$ satisfies the one-dimensional diffusion equation

$$\frac{\partial^2 U}{\partial x^2} - \frac{1}{a^2}\frac{\partial U}{\partial t} = 0,$$

where a is a real constant. If $U(x,t) \to 0$ as $t \to \infty$ for all values of x, use the method of separation of variables to show that a solution is given by

$$U(x,t) = [A\cos(\lambda x) + B\sin(\lambda x)]\exp(-\lambda^2 a^2 t),$$ [8]

where A, B and λ are real constants.

If $U(x,t) = 0$ both at $x = 0$ *and* $x = L$ for all values of t, prove that the general solution is

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(\frac{-n^2\pi^2 a^2 t}{L^2}\right).$$ [4]

b) The Fourier transform of a function $f(t)$ is defined as

$$g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\exp(-\mathrm{i}\omega \mathrm{t})\mathrm{dt}.$$

Write down a formula for the inverse transform $f(t)$. [2]

Evaluate the Fourier transform of the function $f(t)$ specified by

$$f(t) = \exp(-\alpha t), \quad 0 \le t < \infty; \qquad \mathcal{R}]\{\alpha\} > 0,$$
$$f(t) = 0, \quad t < 0.$$ [4]

Hence obtain an integral expression for $\exp(-\alpha t)$ valid for $t > 0$. [2]

4. A generating function, $G(x,h)$, for the Legendre polynomials, $P_l(x)$, is defined by

$$G(x,h) \equiv (1-2xh+h^2)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} P_l(x)h^l; \qquad |h|<1, \quad |x| \leq 1.$$

By expanding $G(0,h)$ in powers of h, show that for all l

$$P_{2l+1}(0) = 0; \qquad P_{2l}(0) = \frac{(-1)^l 1.3.5..(2l-1)}{2^l l!}.$$ [7]

By differentiating $G(x,h)$ with respect to h, obtain the recurrence relation

$$(l+1)P_{l+1}(x) - (2l+1)xP_l(x) + lP_{l-1}(x) = 0; \qquad l \geq 1.$$ [7]

Given that $P_0(x) = 1$ and $P_1(x) = x$, deduce expressions for $P_2(x)$ and $P_3(x)$. [3]

Sketch the functions $P_l(x)$ for $-1 \leq x \leq 1$ and l = 0,1,2 and 3. [3]

5. The function $y(x)$ satisfies the second-order differential equation

$$x\frac{d^2y}{dx^2} + (\beta - x)\frac{dy}{dx} - \alpha y = 0,$$

where α and β are constants and β is not an integer. Show that this equation has two independent solutions, $y_1(x)$ and $y_2(x)$, of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+k}$$

with $k = 0$ and $k = 1-\beta$. [4]

Derive the recurrence relation

$$\frac{a_n}{a_{n-1}} = \frac{(n-1+k+\alpha)}{(n+k)(n+k-1+\beta)}.$$ [4]

Hence show that for $k = 0$,

$$y_1(x) \equiv AF(\alpha,\beta;x) = A\left[1 + \frac{\alpha}{\beta}x + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{x^2}{2!} + \frac{\alpha(\alpha+1)..(\alpha+n-1)}{\beta(\beta+1)..(\beta+n-1)}\frac{x^n}{n!} + ...\right]$$ [5]

and that for $k = 1-\beta$

$$y_2(x) = Bx^{1-\beta}F(\alpha-\beta+1, 2-\beta; x),$$ [4]

where A and B are constants.

If $\alpha = \beta$, what well-known function is $y_1(x)$? [3]

6. The function $f(x)$ is periodic with period 2π and is continuous within the interval $-\pi < x < \pi$. If $f(x)$ has a Fourier expansion of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

then prove that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{dx}; \qquad \mathrm{b_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{f(x)}\sin(\mathrm{nx})\mathrm{dx}.$$ [8]

If $f(x) = x^2$ for $-\pi \le x \le \pi$, evaluate the coefficients a_n and b_n. [6]

Hence, by setting $x = 0$ and $x = \pi$ respectively, show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12},$$ [3]

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$ [3]

7. The divergence theorem states that for any vector field $\mathbf{V}$,

$$\int_\tau \nabla.\mathbf{V}\mathbf{d}\tau = \int_\mathbf{S} \mathbf{V}.\mathbf{dS},$$

where S is a closed surface enclosing the volume τ and $\mathbf{dS} = \hat{\mathbf{n}}\ dS$ where $\hat{\mathbf{n}}$ is a unit vector along the outward normal to S. If ϕ is a scalar function and $\mathbf{v}$ is a vector, prove that

$$\nabla.(\phi\mathbf{v}) = \nabla\phi.\mathbf{v} + \phi(\nabla.\mathbf{v}).$$ [3]

Hence show that if $\mathbf{v} = \nabla\psi$ where ψ is another scalar function,

$$\nabla.(\phi\nabla\psi) = \nabla\phi.\nabla\psi + \phi(\nabla^\mathbf{2}\psi).$$ [1]

Use the results given above to obtain the relation

$$\int_\tau \nabla\phi.\nabla\psi\mathbf{d}\tau + \int_\tau \phi\nabla^\mathbf{2}\psi\mathbf{d}\tau = \int_\mathbf{S} \phi\nabla\psi.\mathbf{dS},$$ [2]

and hence deduce that

$$\int_\tau (\phi\nabla^2\psi - \psi\nabla^2\phi)d\tau = \int_S (\phi\nabla\psi - \phi\nabla\phi)..\mathbf{dS}.$$ [4]

Verify this relation for a sphere of unit radius centred at the origin, and where ϕ and ψ are defined in Cartesian coordinates by

$$\phi = x^2; \qquad \psi = z^4.$$ [10]

PHYS2B72/2000 **END OF PAPER**