Answer SIX questions from Section A and THREE questions from Section B.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

SECTION A

[Part marks]

1. Describe the tunnelling phenomenon of quantum mechanics. [3]
   Briefly discuss a physical process in which it is important. [4]

2. A particle moves in a one-dimensional finite square potential well with boundaries located at $x = -a$ and $x = +a$. Contrast qualitatively the predictions of classical and quantum mechanics for this system including a sketch of the position probability distribution for the lowest quantum energy state in both cases. [7]

3. State the **uncertainty principle** for a particle moving in one dimension with momentum variable p_x and position variable x. [2]
   Define the uncertainties in momentum and position in terms of the corresponding operators and their expectation values, and explain the meaning of these definitions. [5]

4. What is meant by the statement that an energy level is **degenerate**? [2]
   What is meant by the **statistical weight** of such a level? Give the statistical weight of a hydrogen energy level of principal quantum number n. [3]
   State how this statistical weight is modified if electron spin is neglected and explain why. [2]

5. Define the **commutator** of two operators $\hat{A}$ and $\hat{B}$. [2]
   Show that the commutation relation for the operators, $\hat{x}$ and $\hat{p_x} = -i\hbar\frac{\partial}{\partial x}$, representing position and momentum in one dimension is $[\hat{x}, \hat{p_x}] = i\hbar$ [5]

6. The wave function of a particle moving in one dimension is $\Psi(x,t)$. What is meant by the statement that this wave function is **normalised** to unity? [2]
   If the particle is localised between $x = -a$ and $x = +a$ and its space wave function is $u(x) = c\cos\left(\frac{\pi x}{2a}\right)$, determine the normalisation constant, c. [5]

7. Briefly describe Compton's X-ray scattering experiment. [6]
   What is inverse Compton scattering? [1]

8. Describe briefly the Stern-Gerlach experiment for the measurement of the magnetic moments of atoms, and discuss the signficance of the results obtained. [7]

SECTION B

9. Discuss the wave and particle nature of both matter and electromagnetic radiation. Make detailed reference to evidence from the photoelectric effect and the experiment of Davisson and Germer. **[20]**

10. The reduced radial Schrödinger equation, in atomic units, for an electron in a hydrogen atom for which the orbital angular momentum quantum number is zero, is

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} - \frac{1}{\nu^2}\right) F(r) = 0,$$

where ν is related to the total energy by $E = -1/(2\nu^2)$.

(a) Show that as $r \to \infty$, a physically acceptable solution of this equation is

$$F(r) \to e^{-r/\nu}.$$

[4]

(b) By putting $F(r) = \exp(-r/\nu)y(r)$ in the radial Schrödinger equation, show that

$$\frac{d^2y}{dr^2} = \frac{2}{\nu}\left(\frac{d}{dr} - \frac{\nu}{r}\right) y.$$

[5]

(c) Assuming that $y(r)$ can be expanded as the series

$$y(r) = \sum_{p=0}^{\infty} a_p r^{p+1},$$

where $a_0 \neq 0$, show that the coefficients a_p in the series satisfy the recurrence relation,

$$p(p+1)a_p = \frac{2}{\nu}(p - \nu)a_{p-1}.$$

[6]

(d) Solutions of the radial Schrödinger equation exist which are bounded for all r provided that $\nu = n$, where n is a positive integer. Derive the un-normalized radial function for the 4s state. **[5]**

11. The one-dimensional time-independent Schrödinger equation for a particle of mass m moving in a potential $V(x)$ is given by

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right) u(x) = Eu(x),$$

where E is the total energy. A potential barrier is defined by

Region 1	$x < 0$	and	$V(x) = 0$
Region 2	$0 \leq x \leq a$	and	$V(x) = V_0$
Region 3	$x > a$	and	$V(x) = 0$

where $V_0 > 0$ and $a > 0$.
Consider the special case when the particle energy is equal to the barrier height, $E = V_0$. Let there be a flux of particles incident on the barrier from $x \leq 0$.

(a) Demonstrate that the Schrödinger equation has the following solutions in the three regions,

$$\begin{aligned} u_1(x) &= e^{ikx} + Ae^{-ikx} \\ u_2(x) &= Bx + C \\ u_3(x) &= De^{ikx}, \end{aligned}$$

where $k^2 = \dfrac{2mE}{\hbar^2}$ and A, B, C and D are constants. [4]

(b) What is the significance of the two terms in the solution in Region 1 and why is there no term in e^{-ikx} for $x > a$? [2]

(c) State the continuity conditions that must be satisfied by the wave function at $x = 0$ and $x = a$. [2]

(d) Show that the reflection coefficient for the barrier $R = |A|^2$ is given by

$$R = \left[1 + \frac{4}{k^2a^2}\right]^{-1}$$

[8]

(e) What would be the transmission coefficient for this barrier if the incident particles were electrons of energy 0.5 atomic units incident on a barrier of width 2 atomic units of length? [2]

(f) What is the relationship between the de Broglie wavelength of the incident particles and the barrier width if the barrier transmits half of the incident particles? [2]

12. (a) State the eigenvalues of the orbital angular momentum operators $\hat{L}^2$ and $\hat{L}_z$. How are the corresponding quantum numbers related? [3]

(b) If the orbital angular momentum quantum number $\ell = 1$, sketch the possible orientations of the the angular momentum vector $\mathbf{L}$ in the semi-classical vector model and explain the diagram. [4]

(c) The operators $\hat{L}_z$ and $\hat{L}^2$ can be expressed in terms of the spherical polar angles (θ, ϕ) as

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi},$$
$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right].$$

Given the unnormalised spherical harmonic function

$$Y(\theta,\phi) = N\,e^{2i\phi}\sin^2\theta,$$

where N is a constant, show that $Y(\theta, \phi)$ is an eigenfunction of $\hat{L}_z$ and $\hat{L}^2$ and determine the corresponding eigenvalues and angular momentum quantum numbers. [7]

(d) Determine the eigenvalues of the operator $\hat{L}_z^2$ in the general case. [3]

(e) Hence also determine the eigenvalues of the operator $\hat{L}_x^2 + \hat{L}_y^2$. [3]

13. (a) Define the **expectation value** of a dynamical variable represented by an operator $\hat{A}$. [2]

(b) A **Hermitian** operator $\hat{A}$ is defined as one for which, for all normalisable functions f and g,

$$\int f^*\hat{A}g\,d\tau = \int(\hat{A}f)^*g\,d\tau.$$

Show that its expectation value is real. [3]

(c) The time-dependent Schrödinger equation is

$$\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t},$$

where $\hat{H}$ is the Hamiltonian operator. Show that, if $\hat{A}$ is Hermitian, then

$$\frac{\partial}{\partial t} < \hat{A} > = \frac{i}{\hbar} < [\hat{H}, \hat{A}] > + < \frac{\partial\hat{A}}{\partial t} > .$$ [8]

(d) Hence obtain Ehrenfests's Theorem for a particle moving in a potential V

$$\frac{\partial}{\partial t} < \hat{\mathbf{p}} > = - < \nabla V >,$$

where $\hat{\mathbf{p}}$ is the momentum operator. Interpret this result physically. [7]

END OF PAPER