UNIVERSITY OF LONDON
(University College London)

PHYSICS 2B21: Mathematical Methods in Physics and Astronomy

17-MAY-01

All questions may be attempted. *Credit will be given for all work done correctly. Numbers in square brackets show the provisional allocation of marks per sub-section of the question.*

You may find useful the relation between the basis vectors in spherical polar and Cartesian coordinates:

$$\begin{aligned} \underline{\hat{e}}_r &= \sin\theta\cos\phi\,\underline{\hat{e}}_x + \sin\theta\sin\phi\,\underline{\hat{e}}_y + \cos\theta\,\underline{\hat{e}}_z\,, \\ \underline{\hat{e}}_\theta &= \cos\theta\cos\phi\,\underline{\hat{e}}_x + \cos\theta\sin\phi\,\underline{\hat{e}}_y - \sin\theta\,\underline{\hat{e}}_z\,, \\ \underline{\hat{e}}_\phi &= -\sin\phi\,\underline{\hat{e}}_x + \cos\phi\,\underline{\hat{e}}_y\,, \end{aligned}$$

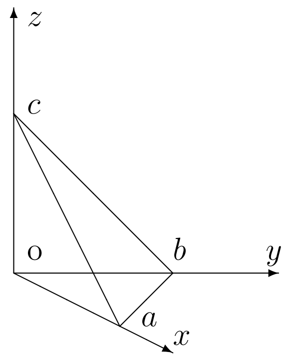
1. (a) By expressing both sides of the equation explicitly in Cartesian coordinates, show that

$$\underline{C} \times (\underline{\nabla} \times \underline{S}) = \underline{\nabla}\,(\underline{C} \cdot \underline{S}) - (\underline{C} \cdot \underline{\nabla})\,\underline{S}\,,$$

where $\vec{S}$ is a vector function of (x, y, z) and $\vec{C}$ is a constant vector. **[6 marks]**

(b) State Stokes' theorem in integral form. **[2 marks]**

Calculate the line integral $I = \oint_\gamma \underline{W} \cdot \underline{ds}$ of the vector **[6 marks]**



$$\underline{W} = (2y^2 - 3z^2)\,\underline{\hat{e}}_x - xz^2\,\underline{\hat{e}}_y - xy^2\,\underline{\hat{e}}_z\,.$$

The closed contour γ is the perimeter of the triangle with vertices $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (0, 0, 1)$ in that order.

Verify Stokes' theorem for the vector $\underline{W}$ by carrying out an integration over the three faces *oab*, *obc* and *oca* of the tetrahedron in the $z = 0$, $x = 0$ and $y = 0$ planes respectively. **[6 marks]**

2. (a) In spherical polar coordinates ($x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$), the line element is given by

$$d\underline{r} = dr\,\underline{\hat{e}}_r + r\,d\theta\,\underline{\hat{e}}_\theta + r\sin\theta\,d\phi\,\underline{\hat{e}}_\phi\ ,$$

where $\underline{\hat{e}}_r$, $\underline{\hat{e}}_\theta$, and $\underline{\hat{e}}_\phi$ are basis vectors in the directions of increasing r, θ and ϕ respectively. Show that in these coordinates

$$\underline{\nabla}\,f = \left(\frac{\partial f}{\partial r}\right)\underline{\hat{e}}_r + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right)\underline{\hat{e}}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial f}{\partial \phi}\right)\underline{\hat{e}}_\phi\ .$$ **[4 marks]**

If $f = x^2 + y^2$, evaluate $\underline{\nabla}f$ in both Cartesian and spherical polar coordinates and show that they are equal in magnitude and direction. **[6 marks]**

(b) The function $u(x,t)$ satisfies the differential equation

$$\left(\frac{\partial^2 u}{\partial t^2}\right) + \alpha^2\,u = c^2\left(\frac{\partial^2 u}{\partial x^2}\right)\ ,$$

where c and α are real constants. By seeking a solution of the equation in the separable form $u(x,t) = X(x)\times T(t)$, find the most general solution for which $u(0,t) = 0$, $u(L,t) = 0$, and $u(x,0) = 0$. **[10 marks]**

3. (a) By considering the action on the basis vectors $\underline{\hat{e}}_x$ and $\underline{\hat{e}}_y$, show that a counter-clockwise rotation in a two-dimensional space may be represented by the matrix

$$\underline{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\ ,$$ **[2 marks]**

and a reflection in a line making an angle α with the x-axis by

$$\underline{A}(\alpha) = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}\ .$$ **[3 marks]**

Demonstrate by matrix techniques that a reflection in a line making an angle α with the x-axis followed by a rotation through an angle 2θ is equivalent to a reflection in a line making an angle $\alpha+\theta$ with the x-axis. **[4 marks]**

What would the combined effect be if the actions of the two operations were interchanged? **[1 marks]**

(b) The matrices $\underline{A}$, $\underline{B}$, and $\underline{D}$ are related by $\underline{D} = \underline{A}\,\underline{B}$. Given that

$$\underline{A} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \quad \text{and} \quad \underline{D} = \begin{pmatrix} 7 & -1 & 0 \\ 3 & 1 & -1 \\ 3 & 9 & 5 \end{pmatrix}\ ,$$

evaluate $\underline{A}^{-1}$. **[7 marks]**

Hence derive the value of $\underline{B}$. **[3 marks]**

4. A real quadratic form F is defined by

$$F = \underline{X}^T \, \underline{A} \, \underline{X} = (x_1, x_2, x_3) \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} .$$

Show that two of the eigenvalues of the matrix $\underline{A}$ are $\lambda_1 = 2$ and $\lambda_2 = 3$ and determine the third one. **[4 marks]**

Derive the three corresponding normalised eigenvectors and show that they are mutually orthogonal. **[8 marks]**

By performing an orthogonal transformation to a new vector $\underline{y}$,

$$\underline{x} = \underline{R} \, \underline{y} \, ,$$

with

$$\underline{R}^T \, \underline{R} = \underline{I} \, ,$$

show that F can be written in the diagonal form

$$F = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 \, . \qquad (*)$$ **[2 marks]**

Express y_1, y_2 and y_3 in terms of x_1, x_2 and x_3. **[4 marks]**

By substituting these expressions for the y_i into equation (*), show that one recovers the original form for F in terms of the x_i. **[2 marks]**

5. Show that the second order differential equation

$$(1 - x^2) \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + m(m+2) y = 0 \, ,$$

where m is a non-negative integer, has two solutions of the form

$$y = \sum_{n=0}^{\infty} a_n \, x^{n+k} \, , \qquad a_0 \neq 0$$

with $k = 0$ or $k = 1$. **[6 marks]**

Find the ratio a_{n+2}/a_n for both series. **[4 marks]**

Show that the series expansion for <u>one</u> of the solutions terminates at $n = m - k$. **[3 marks]**

For $m = 0, 1, 2$, expand $y_m = C_m \sin(m+1)\theta / \sin\theta$ as a polynomial in $x = \cos\theta$. Show that, for C_m constant, the resulting polynomial is a solution of the original differential equation. **[7 marks]**

6. The even function $f(x)$ is periodic with period 2π. In the interval $-\pi < x < \pi$, it is given by

$$f(x) = \begin{cases} \pi + x\,, & \text{if} \quad -\pi < x < -\frac{1}{2}\pi\,, \\ \frac{1}{2}\pi\,, & \text{if} \quad -\frac{1}{2}\pi < x < \frac{1}{2}\pi\,, \\ \pi - x\,, & \text{if} \quad \frac{1}{2}\pi < x < \pi\,, \end{cases}$$

Sketch the function in the above interval. **[1 marks]**

If $f(x)$ is expanded in a Fourier series of the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx\,,$$

show, by using the orthogonality of the cosine functions, that the Fourier coefficients are given by

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\, \cos nx\, dx\,.$$

[5 marks]

Evaluate the coefficients a_n and show that the Fourier series for $f(x)$ is

$$f(x) = \frac{3\pi}{8} + \frac{2}{\pi}\left[\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos[(2n+1)x] - 2\sum_{n=0}^{\infty}\frac{1}{(4n+2)^2}\cos[(4n+2)x]\right].$$

[10 marks]

State Parseval's theorem and apply it to the above series to evaluate

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}\,.$$

[4 marks]

7. The Legendre polynomials $P_n(x)$ may be defined by the generating function

$$g(x,t) \equiv (1 - 2x\,t + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\,t^n \,.$$

By differentiating $g(x,t)$ partially with respect to t or x, derive the recurrence relations:

(a) $$(2n+1)\,x\,P_n(x) = (n+1)\,P_{n+1}(x) + n\,P_{n-1}(x)\,,$$ **[7 marks]**

(b) $$P_n(x) = \frac{dP_{n+1}(x)}{dx} + \frac{dP_{n-1}(x)}{dx} - 2x\,\frac{dP_n(x)}{dx}\,.$$ **[5 marks]**

By differentiating (a) with respect to x, and substituting into (b), show that

(c) $$\frac{dP_{n+1}(x)}{dx} = (n+1)\,P_n(x) + x\,\frac{dP_n(x)}{dx}\,.$$ **[4 marks]**

As $x \to \infty$ the Legendre polynomials behave as

$$P_n(x) \approx \frac{(2n)!}{2^n (n!)^2}\,x^n\,.$$

Show that this behaviour is consistent with relations (a) and (c). **[4 marks]**

END OF PAPER