Queen Mary
University of London

M. Sc. Examination by course unit 2010

ASTM001 Solar System

Duration: 3 hours

Date and time: 4 June 2010, 1430h–1730h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): C. D. Murray, J. R. Donnison

Section A: Each question carries 5 marks. You should attempt ALL five questions and give definitions where appropriate.

Question 1 Describe briefly (in a few sentences) what is meant by each of the following terms:

(a) Hirayama families [2.5]

(b) Hill sphere [2.5]

Question 2 Describe briefly (in a few sentences) what is meant by each of the following terms:

(a) Co-orbital satellites [2.5]

(b) Shepherding satellites [2.5]

Question 3 Describe briefly (in a few sentences) what is meant by each of the following terms:

(a) Synchronous rotation [2.5]

(b) Tidal heating [2.5]

Question 4 Describe briefly (in a few sentences) what is meant by each of the following terms:

(a) Chaotic motion [2.5]

(b) Surface of section [2.5]

Question 5 Describe briefly (in a few sentences) what is meant by each of the following terms:

(a) Trans-Neptunian objects [2.5]

(b) The Nice model [2.5]

Section B: Each question carries 25 marks. There are 4 questions. You may attempt all questions, but only marks for the best 3 questions will be counted.

Question 6 A planet moves under the gravitational attraction of a central star and its resulting path is an ellipse with the star at one focus. The relationship between the planet's radial distance, r from an origin O at the star and its true anomaly, f, is given in polar coordinates by,

$$r = \frac{a(1-e^2)}{1+e\cos f} \ ,$$

where a and e are the semi-major axis and eccentricity of the orbit. The same orbital path can be described by the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \ ,$$

where $b = a\sqrt{1-e^2}$ is the semi-minor axis of the ellipse, (x, y) are the coordinates of the planet in a frame with origin, O', at the centre of the ellipse (midway between the two foci) and the x-axis lies along the line joining the two foci.

(a) Draw a diagram to illustrate the relationship between the polar coordinate system with origin O and the cartesian coordinate system with origin O'. Sketch a circle of radius a centred on the origin O' and use it to illustrate the relationship between f and the eccentric anomaly, E. Derive expressions for $r\cos f$ and $r\sin f$, and hence show that

$$r = a(1 - e\cos E).$$

[9]

(b) Substitute the result from part (a) in the equation

$$\dot{r} = \frac{na}{r}\sqrt{a^2e^2 - (r-a)^2} \ ,$$

where n is the mean motion of the object, and hence solve it to derive Kepler's equation,

$$M = E - e\sin E \ ,$$

where $M = n(t-\tau)$ is the mean anomaly and τ is a constant. [9]

(c) E can be expressed as a power series in e. State two possible limitations to the use of such a series for numerical solutions to Kepler's equation. In planetary dynamics, why is it advantageous to express quantities as series in M rather than E? [7]

Question 7 In the planar circular restricted three-body problem the equations of motion of the test particle in the rotating frame are given by

$$\ddot{x} - 2\dot{y} = U_x \qquad \ddot{y} + 2\dot{x} = U_y \ ,$$

where

$$U = \frac{1}{2}(x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

and $U_x = \partial U/\partial x$, $U_y = \partial U/\partial y$, $\mu_1 = m_1/(m_1 + m_2)$, $\mu_2 = m_2/(m_1 + m_2)$, $\quad m_2 < m_1$. The square of the distances from the particle to the masses m_1 and m_2 are given by $r_1^2 = (x + \mu_2)^2 + y^2$, $r_2^2 = (x - \mu_1)^2 + y^2$ respectively.

(a) Show that the equations of motion have equilibrium solutions at the points given by $x_0 = \frac{1}{2} - \mu_2$, $y_0 = \pm\sqrt{3}/2$. By considering a small displacement (X, Y) from (x_0, y_0), derive a set of simultaneous linearised differential equations of the form

$$\begin{aligned} (D^2 - U_{xx})X - (2D + U_{xy})Y &= 0 \\ (2D - U_{xy})X + (D^2 - U_{yy})Y &= 0 \end{aligned}$$

where $D \equiv d/dt$ and the $U_{xx} = \partial^2 U/\partial x^2$, etc are the partial derivatives evaluated at (x_0, y_0). [12]

(b) The numerical values of U_{xx}, U_{xy} and U_{yy} at (x_0, y_0) are:

$$U_{xx} = \frac{3}{4}, \quad U_{xy} = \pm\frac{3\sqrt{3}}{4}(\mu_1 - \mu_2), \quad U_{yy} = \frac{9}{4} \ ,$$

where the upper sign in U_{xy} is for $y_0 = +\sqrt{3}/2$ and the lower sign for $y_0 = -\sqrt{3}/2$. By assuming solutions of the form $X = \alpha e^{\lambda t}$, $Y = \beta e^{\lambda t}$ where α, β and λ are constants, show that the simultaneous equations have a zero determinant provided

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu_1\mu_2 = 0 \, .$$

Solve this to show that the equilibrium points are linearly stable provided

$$1 - 27\mu_1\mu_2 > 0.$$

[9]

(c) The small Saturnian satellite, Pallene, is embedded in a faint, continuous, narrow ring of material. Sketch the path of a ring particle in this system in a frame moving with the angular velocity of Pallene indicating the location of (x_0, y_0) in the Saturn-Pallene system. Briefly discuss two possible origins of such a ring making clear which you think is more plausible. [4]

Question 8 The tidal potential per unit mass experienced by a satellite of mass m moving in a circular orbit of radius a due to the tidal bulge it raises on a homogeneous planet of radius A and mass M ($\gg m$) is

$$V = -k_2 \frac{\mathcal{G}m}{a} \left(\frac{A}{a}\right)^5 P_2(\cos\theta) \ ,$$

where k_2 (a constant) is the Love number of the planet, $\mathcal{G}$ is the universal gravitational constant, θ is the lag angle and $P_2(x) = \frac{1}{2}(3x^2 - 1)$ is the Legendre polynomial of degree 2.

(a) Calculate the tangential component of the force due to this potential and hence show that the resulting torque experienced by the satellite is

$$\Gamma = \mathcal{G}\frac{m^2}{a}\left(\frac{A}{a}\right)^5 \frac{3}{2} k_2 \sin 2\theta.$$

[6]

(b) Let E be the sum of the rotational energy of the planet and the orbital energy of the satellite–planet system. Show that $\dot{E}$, the rate of change of this energy, is given by

$$\dot{E} = I\Omega\dot{\Omega} + \frac{1}{2} m n^2 a \dot{a} \ ,$$

where I is the moment of inertia of the planet, Ω is the rotational frequency of the planet and n is the mean motion of the satellite. [6]

(c) Use the conservation of the total angular momentum (rotational plus orbital) of the system and the result from part (b) to show that

$$\dot{E} = -\frac{1}{2} m a n \dot{a} (\Omega - n).$$

[7]

(d) Given that $\dot{E} = -\Gamma(\Omega - n) < 0$, use the results from parts (a), (b) and (c) to show that $\dot{a} \propto a^{-11/2}$ for a given satellite, and give the explicit form of the constant of proportionality. By integrating the equation and assuming that the the initial orbital radius is much smaller than the current one, show that this result can be used to provide evidence of significant tidal evolution in a system of satellites orbiting a planet. [6]

Question 9 (a) A planet moves in a circular orbit about a central star. In a rotating frame moving with the uniform velocity of the planet, the path of an unperturbed particle moving on an elliptical orbit interior to the orbit of the planet can appear to be stationary at 'cusp' points in its orbit for certain critical values of its eccentricity, e_c. These occur because at its apoapse the particle's angular velocity exactly matches the (constant) angular velocity of the planet. Given that the angular momentum per unit mass of the particle is $h = na^2\sqrt{1-e^2}$, derive an expression for the angular velocity of the particle at its apoapse. Hence show that if the particle's motion is always in the orbital plane of the planet and the particle is in a $p+1:p$ resonance, show that the value of e_c is given by the solution of the cubic equation

$$\left(\frac{p}{p+1}\right)^2 (1+e_c)^3 - 1 + e_c = 0 \ .$$

[10]

(b) To lowest order in the eccentricity e, the averaged disturbing function experienced by a particle at the $p+1:p$ interior resonance in the planar, circular restricted three-body problem is

$$R = \frac{\mathcal{G}m'}{a'} f(\alpha)\, e \, \cos\varphi \ ,$$

where

$$\varphi = (p+1)\lambda' - p\lambda - \varpi$$

and where $\mathcal{G}$ is the universal gravitational constant, m' and a' denote the mass and semi-major axis respectively of the perturbing object, f is a function of $\alpha = a/a'$ where a is the semi-major axis of the particle, λ' and λ denote the mean longitudes of the perturbing planet and particle respectively, and ϖ denotes the longitude of pericentre of the particle.

Ignoring the variation of the mean longitude at epoch, write down an expression for $\dot{\phi}$ and explain what is meant by *exact resonance*.

Given that

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{-3}{a^2}\frac{\partial R}{\partial \lambda} \ ,$$

and ignoring any precession due to the resonance, show that φ satisfies the pendulum equation,

$$\ddot{\varphi} \approx k \, \sin\varphi \ ,$$

to lowest order in the mass, giving the explicit form of the constant k. Sketch the expected behaviour of ϕ as a function of time when (i) the system is close to exact resonance and (ii) the system is in resonance but close to the separatrix of the motion. [15]

End of Paper