Answer FIVE questions.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of marks per sub-section of a question.

1. (a) If the general quadratic equation is of the form $ax^2 + bx + c = 0$ show by 'completing the square' that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$ [5]

(b) Determine the condition(s) that the general quadratic equation has

(i) two real solutions,

(ii) equal solutions,

(iii) two complex conjugate solutions and

(iv) two imaginary solutions of equal magnitude. [5]

(c) The turning points $r_{min/max} = r_m$ of the orbit of a planet, mass m, about an infinitely massive body placed at the origin of a coordinates system, are determined by the energy conservation equation

$$\frac{1}{2}\frac{L^2}{m}\frac{1}{r_m^2} - \frac{C}{r_m} = E$$

where r_m is the planet's distance from the origin at the turning points and E is the total energy of the system; L and C are constants. Show by completing the square that $\left(\frac{1}{r_m}\right)$ are given by

$$\frac{1}{r_m} = \frac{mC}{L^2} \pm \frac{mC}{L^2}\sqrt{1 + \frac{2EL^2}{mC^2}}.$$ [6]

(d) Determine the value of E where r_{min} and r_{max} are equal. [2]

(e) Determine the value of E for which r_{max} becomes infinite. [2]

2. (a) Show that the sum S_n of the arithmetic series of n terms is given by

$$S_n = a + (a+d) + (a+2d)........ + (a+(n-1)d) = \frac{n}{2}(2a + (n-1)d) \ .$$ [3]

(b) If the first term in an arithmetic series is a and the last (n^{th}) term is l what is the sum of the series? [3]

(c) Show that the sum S_n of the geometric series of n terms is given by

$$S_n = a + ar + ar^2, ..., ar^{n-1} = a\frac{(1-r^n)}{1-r} \ .$$ [3]

(d) At each state in a water purification system a fraction f of the contaminant remaining in the water at that stage is removed. By considering the first few stages demonstrate that after n stages the concentration of the contaminant remaining in the water, r_n is given by

$$r_n = x(1-f)^n$$

and that the contaminant removed, x_n is given by

$$x_n = fx(1 + (1-f) + (1-f)^2 + ...(1-f)^{n-1}) \ .$$

where x is the initial concentration of the contaminant. [5]

(e) Sum the series for x_n and hence show that $x_n + r_n = x$. [6]

3. (a) Write down the definition of the derivative $df(x)/dx$ in terms of a limiting procedure. [4]

(b) Starting from the definition of the differential df of a function of two variables $f(x, y)$ obtain the expression

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \ .$$ [6]

(c) The speed distribution $dN(v)/dv$ in a perfect gas is given by

$$\frac{dN(v)}{dv} = 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} v^2 e^{-\frac{1}{2}mv^2/kT} \ .$$

Obtain an expression for the most probable speed given that at this speed the above distribution is at maximum. [5]

(d) The wavelength distribution of black body radiation $dN/d\lambda$ is given by

$$\frac{dN}{d\lambda} = \frac{8\pi hc}{\lambda^5}\frac{1}{(e^{hc/\lambda kT} - 1)} \ .$$

If $x = hc/\lambda kT$ show that the maximum of the wavelength spectrum is at a value of x such that $x = 5(1 - e^{-x})$. [5]

4. (a) Write down the definition of the definite integral in terms of a limiting procedure of elementary areas. **[3]**

(b) Write down the definition of the indefinite integral. **[3]**

(c) Show that the derivative of an indefinite integral of $f(x)$ is $f(x)$. **[3]**

(d) By considering a circle of radius R to be made up of differential annuli determine the surface area of a circle. **[3]**

(e) By considering a sphere of radius R to be made up of differential spherical shells determine the volume of a sphere. **[3]**

(f) If

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta) \quad \text{where} \quad d\Omega = 2\pi \sin\theta d\theta$$

obtain the total cross section σ_T by integrating the above from $\theta = 0$ to $\theta = \pi$. **[5]**

5. (a) Obtain the Taylor expansion in two variables, as far as the terms shown,

$$f(x,y) = f(a,b) + f_x \Delta x + f_y \Delta y + \frac{1}{2}\left\{f_{xx}\Delta x^2 + 2f_{xy}\Delta x \Delta y + f_{yy}\Delta y^2\right\} \ldots.$$

where $\Delta x = x - a$ and $\Delta y = y - b$. **[5]**

(b) Show that the term involving the second derivatives can be written as

$$\frac{1}{2}\left\{f_{xx}\left(\Delta x + \frac{f_{xy}}{f_{xx}}\Delta y\right)^2 + \Delta y^2\left(f_{yy} - \frac{f_{xy}^2}{f_{xx}}\right)\right\} .$$ **[5]**

(c) From this expression explain why the conditions for a local maximum are $f_{xx} < 0,\ f_{yy} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$. **[3]**

(d) Show that the coordinates of the stationary points of the function

$$f(x,y) = (x^2+y^2)^2 - 2a^2(x^2-y^2)$$

are determined by

$$4x(x^2+y^2-a^2) = 0 \quad , \quad 4y(x^2+y^2+a^2) = 0$$

and are $(0,0)$, $(a,0)$ and $(-a,0)$. **[4]**

(e) Determine the nature of the stationary point $(0,0)$. **[3]**

6. (a) Define the scalar product of two vectors $\vec{A}$ and $\vec{B}$ and from the definition obtain an expression for the scalar product in cartesian coordinates. **[4]**

(b) Define the vector product of two vectors $\vec{A}$ and $\vec{B}$ and from the definition obtain an expression for the vector product in cartesian coordinates. **[5]**

(c) Two charged particles separated by $\vec{r}$ and travelling with velocities $\vec{v_1}$ and $\vec{v_2}$ exert forces proportional to $\vec{v_1} \times (\vec{v_2} \times \vec{r})$ and $\vec{v_2} \times (\vec{v_1} \times (-\vec{r}))$ on each other.

What is the condition that the sum of these forces is zero? **[5]**

(You may use without proof that

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \text{ .)}$$

(d) A pilot is required to fly a plane due East from a point A to another point B, a distance D apart, and then to return due West to A. The speed of the plane relative to the air is u.

If the wind is blowing from East to West with speed v where $v < u$ determine the total time for the journey. **[3]**

If the wind is blowing from the North to the South with speed v where $v < u$ determine the total time for the journey. **[3]**

7. (a) From the Maclaurin expansion

$$f(x) = f(0) + \frac{df(0)}{dx}x + \frac{1}{2!}\frac{d^2 f(0)}{dx^2}x^2 + \frac{1}{3!}\frac{d^3 f(0)}{dx^3}x^3 + \ldots + \frac{1}{n!}\frac{d^n f(0)}{dx^n}x^n$$

obtain the series for

$$e^x, \ \cos x \ \text{ and, by whatever method, } \sin x \text{ .}$$ **[6]**

(b) Use these series to derive the Euler relation

$e^{i\theta} = \cos\,\theta + i\sin\,\theta$ and hence obtain de Moivre's theorem $e^{in\theta} = \cos n\theta + i\sin n\theta$. **[4]**

(c) Evaluate $\ln(1+i)$. **[4]**

(d) Evaluate, in terms of $\sin\theta_i$ and $\cos\theta_i$,

$$E = Re\left[E_0(e^{i(\omega t - \vec{k}\cdot\vec{r})}\right]$$

where **[6]**

$$\vec{k}\cdot\vec{r} = |\vec{k}|x\sin\theta_t - |\vec{k}|y\cos\theta_t \ \ , \ \ \sin\theta_t = n\sin\theta_i \ \text{ and } \ \sin\theta_t > 1 \ .$$