Queen Mary
University of London

M.Sc. EXAMINATION

ASTM003 Angular Momentum and Accretion Processes in Astrophysics

Thursday, 5 May 2005
10:00 – 11:30

Time Allowed: 1h 30m
This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section. Marks awarded are shown next to the questions.
Calculators are NOT permitted in this examination.

SECTION A

You should attempt ALL questions. Marks awarded are shown next to each question.

1. (a) What mechanisms might be responsible for angular momentum transport in an accretion disc? [3 marks]

(b) An accretion disc has a number density $n = 3 \times 10^{21}\ \mathrm{m}^{-3}$ of atoms and ions in its central plane at a distance $R = 0.5\,R_{disc}$ from the central star, where $R_{disc} = 10^{12}$ m is the overall radius of the disc. The temperature at this point is $T = 5 \times 10^4$ K.
Given that the cross-section for collisions between its atomic particles is $\sigma = 10^{-20}\,\mathrm{m}^2$ and their mean free path is given by $L = 1/n\sigma$, what is the mean free path for these particles?
If the speed of sound in the gas is $c_s = \sqrt{\mathcal{R}T/\mu}$, where $\mathcal{R} = 8.3\ \mathrm{J\,mol^{-1}\,K^{-1}}$ is the gas constant, T is the temperature, and μ is the mean molecular mass, estimate the kinematic viscosity ν due to atomic/molecular interactions to within an order of magnitude.
If the evolutionary timescale of an accretion disc is $\tau_{ev} \simeq R_{disc}^2/3\nu$, where R_{disc} is the overall radius and ν is the total kinematic viscosity, estimate to within an order of magnitude the timescale of variations in density of the accretion disc if viscosity is due entirely to atomic/molecular interactions. How does this compare with the timescales observed for the outbursts of dwarf novae? [9 marks]

(c) What is the *alpha model* of viscosity that is used in modelling accretion discs? [3 marks]

[Total 15 marks for question]

2. (a) Derive the equation of hydrostatic equilibrium in the direction z perpendicular to an accretion disc around a star of mass M,

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z} = -\frac{G\,M}{(R^2+z^2)^{3/2}}\,z\,,$$

on the assumption that the mass of the disc is negligible compared with that of the star, where $\rho(R,\phi,z)$ is the density at a point in a cylindrical coordinate system (R,ϕ,z) centred on the star, $P(R,\phi,z)$ is the gas pressure, and G is the constant of gravitation.

Representing the pressure by the ideal gas law, $P = \mathcal{R}\rho T/\mu$, where $\mathcal{R}$ is the gas constant, μ is the mean molecular mass and T is the absolute temperature, solve the equation of hydrostatic equilibrium for an isothermal thin ($z \ll R$) Keplerian accretion disc to show that the density ρ at a distance z from the central plane of the disc is

$$\rho(R,\phi,z) = \rho_0(R,\phi)\,\exp(-z^2/2H^2)\,,$$

where $H(R,\phi) = \sqrt{\mathcal{R}T/\mu\Omega^2}$ is the half thickness and $\Omega(R)$ is the angular velocity at radius R. [14 marks]

(b) Obtain an expression for the surface mass density $\Sigma(R)$ at a radial distance R from the central star for the isothermal accretion disc above. The standard result

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$

may prove useful. [4]

(c) If the sound speed is $c_s = \sqrt{\mathcal{R}T/\mu}$, what is the relationship between c_s, the half thickness H and the angular velocity Ω for the same isothermal disc? [2]

[Total 20 marks for question]

3. (a) The drag force on a spherical particle moving through a gas when the radius a of the particle is smaller than the mean free path of gas molecules can be represented by $F_{drag} = \pi a^2 \rho\, c_s u$, where ρ is the density of the gas, c_s is the speed of sound through the gas, and u is the velocity of the particle relative to the gas.
Derive from this the equation of motion for a spherical dust grain settling under gravity to the central plane of a non-turbulent Keplerian protoplanetary disc,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{3\rho\, c_s}{4\rho_{gr}\, a}\, v - \Omega^2 z \,,$$

where z is the distance from the plane, t is time, $v = \mathrm{d}z/\mathrm{d}t$ is the velocity of the grain relative to the gas, ρ is the density of the gas, ρ_{gr} is the density of the material of the grain, and $\Omega(R)$ is the angular velocity of the disc at a distance R from the the central star. Assume that $z \ll R$. [8 marks]

(b) Assuming that the dust particles in question (a) fall from rest at a height above the disc midplane equal to the half-thickness H of the disc, and that they quickly reach their terminal velocity, show that the terminal velocity v_t is

$$v_t \simeq \frac{4a\rho_{gr}\Omega^2 H}{3\rho c_s} \sim \frac{8a\rho_{gr}\Omega H}{3\Sigma}\,,$$

where Σ is the local surface density. [5]
Hence obtain an approximate expression for the settling time of the dust grains. [2]

[Total 15 marks for question]

 [Next section overleaf.]

SECTION B

Each question carries 50 marks. You may attempt all questions but only marks for the best question will be counted.

1. (a) A binary star system consists of stars of masses m_1 and m_2 moving in circular orbits about their centre of mass. Their separation is D. The system is represented by a Cartesian coordinate system (x, y, z) rotating with an angular velocity Ω so that the stars are stationary at $(x_1, 0, 0)$ and $(x_2, 0, 0)$ respectively (with $x_1 > 0$ and $x_2 < 0$). The x–y plane corresponds to the orbital plane and the origin is at the centre of mass. Show that the gravitational potential Φ in the x–y plane in this rotating coordinate system is

$$\Phi(x,y) \;=\; -\frac{G\,m_2}{\sqrt{y^2+(x-x_2)^2}} \;-\; \frac{G\,m_1}{\sqrt{y^2+(x-x_1)^2}} \;-\; \frac{1}{2}\,(x^2+y^2)\,\Omega^2\;,$$

where G is the constant of gravitation.
Sketch the contours of this gravitational potential in the x–y plane, for the case where the mass of one component is larger than the other. Include the Roche lobe and the position of the L_1 Lagrangian point in this diagram. [10 marks]

(b) Using the expression for Φ given above, show that there exists a stationary point in the potential on the line joining the stars (i.e. for $y = 0$ and $z = 0$), between the two stars, at a distance r_L from the star of mass m_2 (at $x = x_2 + r_L$) where r_L satisfies

$$-\frac{m_1}{m_2}D^3 r_L^2 \;+\; D^3(D-r_L)^2 \;-\; \left(\frac{m_1}{m_2}+1\right) r_L^3(D-r_L)^2 \;+\; \frac{m_1}{m_2}D r_L^2(D-r_L)^2 \;=\; 0$$

Can any precise simple analytic expression be given for this position of the L_1 Lagrangian point in terms of m_1, m_2 and D? [12]

(c) Is any account taken of the Coriolis force if the potential Φ alone is used to calculate the force acting on gas in the system? Give a reason for your answer. [3]

(d) Explain under what circumstances Roche lobe overflow can occur. [3]

(e) Explain the difference between detached, semi-detached and contact binary stars. [3]

(f) Show that in a semi-detached binary, gas from a lobe-filling star of mass m_1 would form a ring of radius

$$R_{ring} \;=\; \frac{(m_1+m_2)}{m_2}\,\frac{r_L^4}{D^3}\;,$$

around a star of mass m_2 if gas interactions were not important, where D is the separation of the two stars.
Hence show that $R_{ring} = D/8$ if the stars have equal mass. [10]

 [This question continues overleaf . . .]

(g) What constraint on the second star does the size of this ring impose for the formation of an accretion disc when the lobe-filling object is a main sequence star? What two types of star might be the components of a cataclysmic variable system? [4]

(h) In practice, viscosity causes the gas to form a disc instead of a ring. Sketch the appearance of a typical accretion disc in a semi-detached binary system. [5]

2. (a) Estimate the amount of potential energy released when mass falls on to a solar mass neutron star as a fraction of the mass energy $E = mc^2$ that is contained in the same material. Assume the radius of the neutron star is 10^4 m. Give your answer accurate to within a factor of 2 if $G = 6.7 \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}$, $c = 3.0 \times 10^8$ m, and the solar mass is 2×10^{30} kg.
Is this larger or smaller than the energy available from nuclear fusion? [4 marks]

(b) Accretion of matter on to neutron stars and black holes may occur up to a maximum rate known as the Eddington limited accretion rate. Explain briefly the physical origin of this limit. [4]

(c) The radiative flux F within an optically thick gas is related to the temperature gradient $\mathrm{d}T/\mathrm{d}r$ over radial distance r from the radiation source by

$$F = -\frac{4ac}{3\kappa\rho} T^3 \frac{\mathrm{d}T}{\mathrm{d}r} ,$$

where T is the temperature, κ is a mean opacity, ρ is the density of the gas, a is the radiation constant, and c is the velocity of light. The radiation pressure within a black body of temperature T is $P_{rad} = aT^4/3$.
Derive from these the expression for the Eddington limited accretion rate, $\dot{m}_{Edd}$, of optically thick material on to a compact object of mass M,

$$\dot{m}_{Edd} = \frac{4\pi c R_c}{\kappa} ,$$

where R_c is the radius of the compact object and G is the gravitational constant. Hence show that if the radius R_c is five times the Schwarzschild radius $R_S = 2GM/c^2$,

$$\dot{m}_{Edd} = \frac{40\pi GM}{c\kappa} .$$

[22]

(d) The dominant source of opacity in a hot plasma is Thomson scattering by electrons which has $\kappa = 0.04\ \mathrm{m^2\ kg^{-1}}$. Estimate, to within an order of magnitude, the maximum mass accretion rate on to a 10^8 solar mass black hole in an active galactic nucleus in units of solar masses per year. You may use 1 year $= 3.2 \times 10^7$ s. [4]

(e) The effective temperature T_{eff} at a distance R from the central object in an accretion disc is approximately given by

$$T_{eff} \simeq 1.2 \times 10^7\ \mathrm{K} \left(\frac{\dot{m}}{M_\odot/\mathrm{yr}}\right)^{1/4} \left(\frac{M}{M_\odot}\right)^{1/4} \left(\frac{R}{10^7\mathrm{m}}\right)^{-3/4} ,$$

where $\dot{m}$ is the mass accretion rate and M is the mass of the central object. Make an order of magnitude estimate of the effective temperatures encountered for Eddington-limited accretion on to a 10^8 solar mass black hole if a typical radius within the disc is $R \sim 10^{12}$ m. [4]

(f) In which part of the electromagnetic spectrum would this radiation be emitted? Comment on how this compares with observations of the radiation from the inner regions of active galactic nuclei and how any discrepancies are explained. [4]

(g) How does the predicted temperature of the accretion disc around a supermassive black hole in an active galactic nucleus compare with the temperatures of accretion discs around stellar mass black holes in binary star systems? [2]

(h) What besides the Eddington limit determines the accretion rate on to the central black hole in an active galactic nucleus? [2]

(i) Sketch as a function of frequency the form of the spectrum of an optically thick accretion disc, labelling the main features. [4]

3. (a) The Navier-Stokes equation which describes the flow of a viscous fluid is

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} ,$$

for each dimension i ($i = 1, 2, 3$) where summation over the repeated index is assumed, and x_i is the ith component of the position vector, v_i is the ith component of the velocity of the fluid, and ρ is the density. σ_{ij} is the stress tensor,

$$\sigma_{ij} = -P\,\delta_{ij} + \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{v}\,\delta_{ij}\right) ,$$

where P is the pressure and η is the coefficient of viscosity.
By neglecting the pressure contribution and using $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$, derive from this the expression for the rate of viscous dissipation per unit volume

$$\epsilon = 2\,\rho\,\nu\,e_{ij}\,e_{ij} ,$$

by neglecting the pressure contribution, where $\nu = \eta/\rho$ is the kinematic viscosity and

$$e_{ij} \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) ,$$

is the rate of strain tensor. [25 marks]

(b) Using the fact that

$$e_{ij}\,e_{ij} \;=\; R^2 \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2$$

for a Keplerian accretion disc, show that the viscous dissipation per unit area is

$$\epsilon_D \;=\; R^2\,\nu\,\Sigma \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2 ,$$

where R is the distance from the central star, Σ is the surface density, and $\Omega(R)$ is the angular velocity about the central star. [5]

(c) Hence show that in a steady state Keplerian disc,

$$\epsilon_D \;=\; \frac{9}{4}\,\Omega^2\,\nu\,\Sigma\,.$$

[4]

(d) The product of the kinematic viscosity and surface density at a radial distance R from a non-rotating star of radius R_* is

$$\nu\,\Sigma \;=\; \frac{\dot{m}}{3\pi}\left[1-\left(\frac{R_*}{R}\right)^{\frac{1}{2}}\right] ,$$

where $\dot{m}$ is the mass accretion rate in a steady state disc. Derive from this the expression for the effective temperature T_{eff} of the accretion disc at a radius R from the central star in a steady state Keplerian disc

$$T_{eff} \;=\; \left[\frac{3GM\dot{m}}{8\pi\sigma R^3}\left(1-\left(\frac{R_*}{R}\right)^{\frac{1}{2}}\right)\right]^{\frac{1}{4}} ,$$

where M is the mass of the star and σ is the Stefan-Boltzmann constant. [10]

(e) Using the fact that the half-thickness of an accretion disc $H \simeq c_s/\Omega$, where c_s is the sound speed and Ω the angular velocity at a radius R, obtain an approximate expression for the half-thickness at a radius R in terms of the mass accretion rate $\dot{m}$ and the mass of the central star if $c_s = \sqrt{\mathcal{R}T/\mu}$ and $R \gg R_*$. Here T is the temperature, $\mathcal{R} = 8.3\ \mathrm{J\,mol^{-1}\,K^{-1}}$ is the gas constant, and μ the molecular mass. Hence make an order of magnitude estimate of the half-thickness at a distance $R = 1\mathrm{AU} = 1.5\times10^{11}\,\mathrm{m}$ from a central star of mass $1M_\odot = 2\times10^{30}\,\mathrm{kg}$ if the disc has an accretion rate of $\dot{m} = 10^{-8}\,M_\odot\,\mathrm{yr}^{-1} = 6\times10^{14}\,\mathrm{kg\,s^{-1}}$ and $R \gg R_*$. You may use $G = 6.7\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ and $\sigma = 5.7\times10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$.
Hence estimate the disc aspect ratio H/R. [6]

[End of examination paper.]